# Hyperreflection groups

David G Radcliffe

Department of Mathematics
Century College

david.radcliffe@century.edu

June 4, 2009

# Objective

- My objective is to generalize the definition of Coxeter group to include a wider class of groups. I call these groups hyperreflection groups.
- These groups are similar enough to Coxeter groups that many of the techniques in Coxeter group theory can be adapted to hyperreflection groups.
- At the same time, the definition is general enough to include other interesting groups, such as graph products and (I think) Artin groups.
- Just as a Coxeter group is generated by reflections, a hyperreflection group is a group that is generated by hyperreflections.

# What is a reflection?

Before we define hyperreflection, we need a suitable definition of reflection.

A reflection of a connected space is an involution (automorphism of order two) whose fixed set separates the space into two components which are interchanged by the involution.

Examples:

- A linear map from $\mathbb{R}^n$ to itself is a reflection if and only if it has the eigenvalues 1 (with multiplicity $n - 1$) and $-1$ (with multiplicity 1).
- Reflection through the origin in the plane does not satisfy our definition, since the origin does not separate the plane.
- The word “space” may potentially refer to any sort of geometric object for which a notion of connectivity can be meaningfully defined.

# Coxeter groups

- Let $W$ be a group generated by a set $S$. The Cayley graph of $(W, S)$ is a graph with vertex set $W$ and edge set $\{\{w, ws\} : w \in W, s \in S\}$.

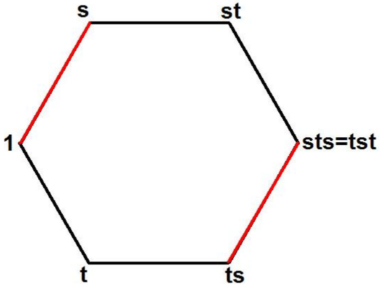
- $W$ acts on $\text{Cay}(W, S)$ by left multiplication.

- $(W, S)$ is a Coxeter system iff every $s \in S$ acts by reflection on $\text{Cay}(W, S)$.

- We say that $W$ is a Coxeter group if there exists a subset $S$ of $W$ so that $(W, S)$ is a Coxeter system.

# Example of a Coxeter system

The dihedral group of order 6 is a Coxeter group. It is generated by two reflections $s$ and $t$ which satisfy $sts = tst$.




The reflection $s$ fixes the two red edges. Removing these edges separates the graph into two components which are interchanged by $s$.

# Cayley hypergraph

A hypergraph is a pair of sets $(V, E)$ such that each element of $E$ is a subset of $V$. Elements of $V$ are called vertices, and elements of $E$ are called edges.

A hypergraph is a graph iff every edge has exactly two elements.

Let $W$ be a group and let $\mathcal{S}$ be a collection of subsets of $W$.

The Cayley hypergraph $\mathrm{Cay}(W, \mathcal{S})$ is the hypergraph whose vertex set is $W$ and whose edge set is $\{wR : w \in W, R \in \mathcal{S}\}$.

# Hyperreflections

Let $X$ be a connected space, and let $R$ be a group of automorphisms of $X$. We say that $R$ is a hyperreflection of $X$ if the following conditions are satisfied.

- $X \setminus \mathrm{Fix}(R)$ is disconnected.
- For any two components $C$ and $D$ of $X \setminus \mathrm{Fix}(R)$, there is a unique $r \in R$ so that $r \cdot C = D$. In other words, the action of $R$ on the components is simply transitive.

If $R$ has order two, then the non-identity element of $R$ is a reflection.

Note: $\mathrm{Fix}(R) = \{x \in X : \forall r \in R, r \cdot x = x\}$

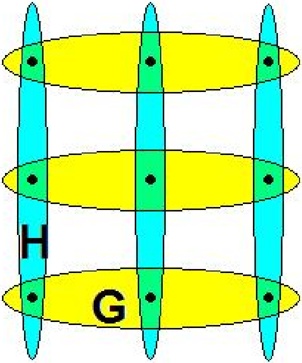
# Hyperreflection systems

Let $W$ be a group, and let $\mathcal{S}$ be a collection of subgroups of $W$ whose union generates $W$. Then $(W, \mathcal{S})$ is a hyperreflection system iff each $R \in \mathcal{S}$ acts as a hyperreflection on $\mathrm{Cay}(W, \mathcal{S})$.

We say that $W$ is a hyperreflection group if there exists a collection $\mathcal{S}$ of two or more subgroups of $W$ so that $(W, \mathcal{S})$ is a hyperreflection system. Note: This reduces to the definition of a Coxeter system in case each group in $\mathcal{S}$ is cyclic of order two.

# Example of a hyperreflection system

If $G$ and $H$ are nontrivial groups, then $(G \times H, \{G, H\})$ is a hyperreflection system.



# Basic properties of hyperreflection groups

We will assume for the remainder of the presentation that $(W, \mathcal{S})$ is a hyperreflection system.

- If $R$ is a subgroup of $W$, then $R$ is a hyperreflection on $\text{Cay}(W, \mathcal{S})$ iff $\exists A \in \mathcal{S}, w \in W$ so that $R = wAw^{-1}$.
- If $R$ and $T$ are hyperreflections then either $R = T$ or $R \cap T = \{1\}$.

# Length function and reduced words

A word is a finite sequence $\vec{r} = (r_1, \ldots, r_n)$ so that $r_i \in \cup\mathcal{S}$ and $r_i \neq 1$ for all $1 \leq i \leq n$. We say that $n$ is the length of the word, and that $\vec{r}$ represents the product $r_1 r_2 \ldots r_n$.

A reduced word for $w$ is a word of minimal length that represents $w$. The length of $w$ is equal to the length of a minimal word that represents $w$.

# The hyperreflections determined by a word

Any word $\vec{r} = (r_1, \ldots, r_n)$ determines another sequence $(t_1, \ldots, t_n)$ defined as follows:

$$
\begin{aligned}
t_1 &= r_1 \\
t_2 &= r_1 r_2 r_1^{-1} \\
t_3 &= r_1 r_2 r_3 r_2^{-1} r_1^{-1} \\
&\ldots
\end{aligned}
$$

Note that $t_k$ is an element of the hyperreflection $xRx^{-1}$, where $x = r_1 \ldots r_{k-1}$ and $r_k \in R \in \mathcal{S}$.

Also, $r_1 \ldots r_k = t_k \ldots t_1$.

# Key results

- The word $\vec{r}$ is reduced if and only if no two of the $t_i$ belong to the same hyperreflection.
- If $\vec{r}$ and $\vec{r'}$ are two reduced words for $w$, then $\{s_1, \ldots, s_n\} = \{s'_1, \ldots, s'_n\}$ and $\{t_1, \ldots, t_n\} = \{t'_1, \ldots, t'_n\}$.
- If $\mathcal{A} \subset \mathcal{S}$ then $(\langle \cup \mathcal{A} \rangle, \mathcal{A})$ is a hyperreflection system.
- If $\mathcal{A} \subset \mathcal{S}$ and $\mathcal{B} \subset \mathcal{S}$ then $\langle \cup (\mathcal{A} \cap \mathcal{B}) \rangle = \langle \cup \mathcal{A} \rangle \cap \langle \cup \mathcal{B} \rangle$
- If $G$ is a graph product with vertex groups $\{V_i\}$, then $(G, \{V_i\})$ is a hyperreflection system.

# Many Tanks!


http://sites.google.com/site/hyperreflection/