Review Questions for the Final Exam in Math 1271

Revised 20 April 2008

1. The displacement in feet of a certain particle moving in a straight line is given by $s = t^2 - 3t - 10$.

   (a) Find the average velocity of this particle over the time interval $0 \le t \le 5$ and over the time interval $2 \le t \le 6$.

   (b) Find the instantaneous velocity of this particle when $t = 2$ and when $t = 5$.

2. Evaluate the following limits without using L'Hospital's rule.

   (a) $\lim_{x\to 3} \frac{x^2 + 2x - 15}{x - 3}$

   (b) $\lim_{x\to 2} \frac{(x+5)(x^2 + 2x - 8)}{x - 2}$

3. Given the function $f(x) = x^3 + 8x^2 + 11x$, find the difference quotient $\frac{f(x) - f(3)}{x - 3}$. Use the difference quotient to find $f'(3)$.

4. Given the function $f(x) = \frac{x}{x+5}$, find the difference quotient $\frac{f(x) - f(2)}{x - 2}$. Use the difference quotient to find $f'(2)$.

5. Evaluate the following limits.

   (a) $\lim_{x\to\infty} \frac{(2x-5)(3x+1)}{(4x+3)(6x+5)}$

   (b) $\lim_{x\to\infty} \frac{(4-3x)(8+5x)}{(2+x)(7+4x)}$

6. Evaluate the limit

$$\lim_{x\to\infty} \frac{(5+7x)(10+3x)}{(5-2x)(9+8x)}$$

7. The volume of a sphere is $V = (4/3)\pi r^3$.

   (a) Find the average rate of change of volume V with respect to radius r when r changes from 5 to 8.

   (b) Find the instantaneous rate of change of V with respect to r when $r = 8$.

8. Find the equation of the tangent line to the curve $y = \sqrt{7x+1}$ at the point where $x = 5$.

9. Suppose the function $f(x)$ is defined as $f(x) = g(x)(x^3 + h(x))$. If

$$\begin{array}{llll} g(-1) = -3 & g'(-1) = 4 & h(-1) = 7 & h'(-1) = 2 \\ g(2) = 5 & g'(2) = 11 & h(2) = -6 & h'(2) = 7 \end{array}$$

   find $f(2)$, $f'(-1)$, and $f'(2)$.

10. Suppose $f(x) = \frac{x^3 - 2x}{g(x)}$. If $g(2) = 8$, $g'(2) = 6$, $g(5) = 10$, and $g'(5) = 15$, find $f(2)$, $f'(2)$, and $f'(5)$.

11. Find $\frac{dy}{dx}$ for the following functions.

(a) $y = (\tan 4x)(\sin 3x + \sec 5x)$

(b) $y = \dfrac{x + \sin 5x}{1 + \cos 4x}$

12. If $f(x) = (5 + 4\sqrt{x})^{3/2}$, find $f'(x)$. If $g(x) = (\cos 3x + \sin 5x)^4$, find $g'(x)$.

13. If $f(x) = \tan(3x^2)$, find $f'(x)$. If $g(x) = \sqrt{1 + \sin 5x}$, find $g'(x)$.

14. Let $f(x) = xg(x^2)$. Find $f'(3)$ given that

$g(3) = 5$	$g(6) = 7$	$g(9) = 10$
$g'(3) = 8$	$g'(6) = -4$	$g'(9) = -6$

15. Let $f(x) = x^3 + g(5x^2 + 2)$. Find $f'(2)$ given that

$g(2) = 8$	$g(20) = -11$	$g(22) = 5$
$g'(2) = -6$	$g'(20) = 4$	$g'(22) = 9$

16. Suppose that y is defined by the following equation.

$$y^3 + 3x^2y^2 = 4x^2 + 16$$

Find $\dfrac{dy}{dx}$. Evaluate $\dfrac{dy}{dx}$ at the point $(1, 2)$.

17. Problem 28 page 213 of the text has the graph of an astroid whose equation is

$$x^{2/3} + y^{2/3} = 4$$

(a) How many continuous functions $y = f(x)$ are defined on the interval $-8 \le x \le 8$ by the equation $x^{2/3} + y^{2/3} = 4$?

(b) Find the equations of all the tangent lines to this curve at all points where $x = 1$.

18. If $f(x) = x^3 \ln(x^2 + 4)$, find $f'(x)$.

19. Two highways meet at right angles in a certain town. Both highways are exact straight lines. Two cars leave the intersection of the highways at the exact same time. One car is traveling east at 50 mph. The other car is traveling north at 70 mph. At 3 hours after the time they leave the intersection, at what rate is the distance between the two cars increasing?

20. A very long ladder 100 ft long rests against a vertical wall. The bottom of the ladder slides away from the wall at a rate of 5 feet per second. How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 60 feet from the wall?

21. A streetlight is mounted at the top of a 28 foot pole. A man 7 feet tall walks away from the pole with a speed of 5 ft/sec along a straight path. How fast is the length of his shadow changing when he is 50 feet from the pole?

22. Let $y = x^{3/2}$.

(a) Find the differential dy when $x = 25$ and $dx = 2.04$.

(b) Find $\Delta y = f(x + \Delta x) - f(x)$ when $x = 25$ and $\Delta x = 2.04$.

23. Find the linear approximation for $f(x) = x^{3/2} + x^{1/2}$ at the point $x = 100$. Use this linearization to find an approximate value of $f(98.4)$.

24. Let $h(x) = xe^{-2x}$. Find the values of x that give the absolute maximum and absolute minimum values of $f(x)$ on the interval $0 \le x \le 3$.

25. Find the absolute maximum value and the absolute minimum value of $f(x) = 40 + 36x + 6x^2 - x^3$ on the interval $-3 \le x \le 4$.

26. The formula in the mean value theorem is $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. Find all numbers c that satisfy this formula where $f(x) = x^3 + 5x^2 + 8x$ on the interval $1 \le x \le 4$.

27. Sketch the graph of a continuous function satisfying all of the following conditions.

    (a) $f'(-2) = 0$, $f'(4) = 0$

    (b) If $x < -2$ then $f'(x) < 0$. If $-2 < x < 4$ or $x > 4$ then $f'(x) > 0$.

    (c) If $x < 1$ or $x > 4$ then $f''(x) > 0$. If $1 < x < 4$ then $f''(x) < 0$.

28. Find the critical points for the following function and classify them as either local maximum points or local minimum points using the first derivative test.

$$f(x) = 10 + 60x + 9x^2 - 2x^3$$

29. Find the critical points for

$$f(x) = 2x^3 + 3x^2 - 36x + 17$$

    Use the second derivative test to classify them as local maximum points or local minimum points.

30. Consider the function $f(x) = xe^{-x/5}$. For what values of x is this function decreasing? For what values of x is the graph of this function concave downward?

31. Sketch the graph of a function $f(x)$ which is continuous and has the following properties.

    (a) $f'(x) > 0$ if $x < -3$ or $x > 5$, and $f'(x) < 0$ if $-3 < x < 5$.

    (b) $f''(x) > 0$ if $x < -8$ or $1 < x < 10$, and $f''(x) < 0$ if $-8 < x < 1$ or $x > 10$.

32. Sketch the graph of a function $f(x)$ which is continuous and has the following properties.

    (a) $f'(x) > 0$ when $x < -3$ or $0 < x < 3$. Also $f'(x) < 0$ when $-3 < x < 0$ or $x > 3$.

    (b) $f''(x) > 0$ when $x > -3$ and $x \neq 3$. Also $f''(x) < 0$ when $x < -3$.

33. Suppose $\left\{ \begin{array}{lll} f(0) = 5 & f(5) = 0 & f(10) = 12 \\ f'(0) = 0 & f'(5) = 0 & f'(10) = 0 \\ f''(0) = -4 & f''(5) = 8 & f''(10) = -7 \end{array} \right\}$. Does this function $f(x)$ have a local maximum or a local minimum at the critical points $x = 0$, $x = 5$, and $x = 10$? Explain your answer.

34. The function $f(x)$ is continuous with a continuous derivative and it satisfies the following conditions:

    (a) $f'(-5) = 0$, $f'(3) = 0$, $f'(8) = 0$

    (b) $f'(x) < 0$ when $x < -5$ or $3 < x < 8$. Also $f'(x) < 0$ when $-5 < x < 3$ or $x > 8$.

    Which of the critical points give a local maximum and which give a local minimum? Justify your answer using the first derivative test.

35. The function $f(x)$ is continuous and satisfies the following conditions.

    (a) $f'(-4) = 0$, $f'(1) = 0$, $f'(6) = 0$, $f''(-1) = 0$ and $f''(3) = 0$.

    (b) $f'(x) > 0$ when $x < -4$ or $1 < x < 6$. Also $f'(x) < 0$ when $-4 < x < 1$ or $x > 6$.

    (c) $f''(x) > 0$ when $-1 < x < 3$. Also $f'(x) < 0$ when $x < -1$ or $x > 3$.

    What are the critical points? Explain which of these critical points give a local minimum and which give a local maximum. Use the first derivative test, then do it again using the second derivative test.

36. Evaluate the limit

$$\lim_{x \to 0} \frac{x \sin 3x}{1 - \cos 4x}$$

37. A farmer has 1200 feet of fencing and wants to enclose a rectangular area and then divide the area into 4 pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?

38. A cylindrical can without a top is to be made to contain 288π cubic feet of liquid. Material for the bottom costs \$16 per square foot and material for the sides costs \$12 per square foot. Find the dimensions that will minimize the cost of the metal to make the can.

39. The top and bottom margins of a poster are each 6 inches and the side margins are each 4 inches. If the area of the printed material on the poster is fixed at 294 in^2, find the dimensions of the poster with the smallest area.

40. A storage room with a square base is to have a volume of 405 ft^3. Material for the four sides cost 5 dollars per square foot and material for the bottom and top costs 9 dollars per square foot. Find the dimensions of the room such that the cost of the material is a minimum.

41. (a) Show that there is a root of the equation $x^3 - 5x^2 + 7x - 8$ in the interval $3 \leq x \leq 4$.
    (b) Take $x_1 = 3.5$ and apply Newton's method to obtain x_2, x_3, and x_4 as estimates of this root.

42. Use Newton's method to find the three roots of the following equation.

$$x^3 - 3x^2 - 10x + 15 = 0$$

43. Evaluate the Riemann sum for $f(x) = (2 + x)^{-1}$ on the interval $1 \leq x \leq 4$ with 6 subintervals, taking sample points to be the midpoints.

44. Evaluate the following integral by interpreting it in terms of areas.

$$\int_0^3 (2 + \sqrt{9 - x^2})\, dx$$

45. Find $f'(x)$ and $g'(x)$ when

$$f(x) = \int_5^x \sqrt{2t^3 + 4}\, dt \quad \text{and} \quad g(x) = \int_4^{x^2} (t^2 + 8t)^{3/2}\, dt$$

46. Find the area of the region bounded by the curve which is the graph of $f(x) = x^2 - 8x$, the x-axis, and the two lines $x = 2$ and $x = 8$.

47. Find the area of the region bounded by the graph of $f(x) = x^2 - 6x$, the x-axis, and the line $x = 10$. Note that there are two parts to this region.

48. Evaluate the integral

$$\int \frac{x}{\sqrt[3]{2x + 1}}\, dx$$

using the substitution $u = 2x + 1$.

49. Evaluate the following integral using the substitution $4u = 3x$.

$$\int \frac{dx}{16 + 9x^2}$$

50. Convert the integral $\int_0^3 (x^2+4)^{3/2} x\,dx$ into a definite integral involving u by using the substitution $u = x^2 + 4$.

51. Convert the integral $\int_0^3 \frac{x^3}{4+x^2}\,dx$ into a definite integral involving u by using the substitution $u = 4+x^2$.

52. Convert the following integral into a definite integral involving θ by using the substitution $3x = 4\sin\theta$. Simplify the integrand.

$$\int_{2/3}^{2\sqrt{2}/3} \frac{x^3\,dx}{\sqrt{16-9x^2}}$$

53. Find the area of the region bounded by the parabola $y = x^2$ and the line $y = x + 2$.

54. Find the area of the region bounded by the parabola $y = 4x - x^2$ and the line $y = -2x$.

55. Find the volume of the solid generated by rotating about the x-axis the region bounded by the curve $y = 10x - x^2$ and the x-axis.

56. Find the volume of the solid generated by rotating the same region about the y-axis.

57. Find the volume of the solid generated by rotating about the x-axis the region bounded by the curve $y = 3\sqrt{x}$ and the line $y = x$.

58. Find the volume of the solid generated by rotating about the y-axis the region bounded by the curve $y = 6x - x^2$ and the line $y = 2x$.

59. Find the volume of the solid generated by rotating about the line $y = -2$ the region bounded by the curve $y = 6x - x^2$ and the x-axis.

60. Find the volume of the solid generated by rotating about the line $x = -3$ the region bounded by the curve $y = 6x - x^2$ and the x-axis.

Solutions

1. (a) $\dfrac{s(5)-s(0)}{5-0}=2; \quad \dfrac{s(6)-s(2)}{6-2}=5$

   (b) $s'(2)=1$; $s'(5)=7$.

2. (a) $\lim\limits_{x\to 3}\dfrac{x^2+2x-15}{x-3}=\lim\limits_{x\to 3}\dfrac{(x-3)(x+5)}{x-3}=\lim\limits_{x\to 3}x+5=8$

   (b) $\lim\limits_{x\to 2}\dfrac{(x+5)(x^2+2x-8)}{x-2}=\lim\limits_{x\to 2}\dfrac{(x+5)(x-2)(x+4)}{x-2}=\lim\limits_{x\to 2}(x+5)(x+4)=42$

3. $\dfrac{f(x)-f(3)}{x-3}=\dfrac{x^3+8x^2+11x-132}{x-3}=x^2+11x+44$; $f'(3)=3^2+3(11)+44=86.$

4. $\dfrac{f(x)-f(2)}{x-2}=\dfrac{x/(x+5)-2/7}{x-2}=\dfrac{7x-2(x+5)}{7(x+5)(x-2)}=\dfrac{5x-10}{7(x+5)(x-2)}=\dfrac{5}{7(x+5)}$; $f'(2)=\dfrac{5}{49}.$

5. (a) $\lim\limits_{x\to\infty}\dfrac{(2x-5)(3x+1)}{(4x+3)(6x+5)}=\lim\limits_{x\to\infty}\dfrac{(2x-5)(3x+1)/x^2}{(4x+3)(6x+5)/x^2}=\lim\limits_{x\to\infty}\dfrac{(2-5/x)(3+1/x)}{(4+3/x)(6+5/x)}=\dfrac{2\cdot 3}{4\cdot 6}=\dfrac{1}{4}.$

   (b) $\lim\limits_{x\to\infty}\dfrac{(4-3x)(8+5x)}{(2+x)(7+4x)}=\lim\limits_{x\to\infty}\dfrac{(4-3x)(8+5x)/x^2}{(2+x)(7+4x)/x^2}=\lim\limits_{x\to\infty}\dfrac{(4/x-3)(8/x+5)}{(2/x+1)(7+4x)}=\dfrac{(-3)(5)}{(1)(4)}=-\dfrac{15}{4}$

6. $\lim\limits_{x\to\infty}\dfrac{(5+7x)(10+3x)}{(5-2x)(9+8x)}=\lim\limits_{x\to\infty}\dfrac{(5+7x)(10+3x)/x^2}{(5-2x)(9+8x)/x^2}=\lim\limits_{x\to\infty}\dfrac{(5/x+7)(10/x+3)}{(5/x-2)(9/x+8)}=\dfrac{7\cdot 3}{-2\cdot 8}=-\dfrac{21}{16}$

7. $\dfrac{V(8)-V(5)}{8-5}=\dfrac{(4/3)\pi 8^3-(4/3)\pi 5^3}{3}=172\pi$
   $V'(r)=4\pi r^2$, $V'(8)=4\pi 8^2=256\pi$

8. $y'=\dfrac{dy}{dx}=\dfrac{7}{2\sqrt{7x+1}}$; $m=y'(5)=\dfrac{7}{2\sqrt{36}}=\dfrac{7}{12}$; $y=y(5)+m(x-5)$; $y=6+\dfrac{7}{12}(x-5)$; $y=\dfrac{7}{12}x+\dfrac{37}{12}$

9. $f(2)=g(2)(2^3+h(2))=5(8+(-6))=10$
   $f'(x)=g'(x)(x^3+h(x))+g(x)(3x^2+h'(x))$
   $f'(-1)=g'(-1)[(-1)^3+h(-1)]+g(-1)[3(-1)^2+h'(-1)]=4(-1+7)+-3(3+2)=24-15=9$
   $f'(2)=g'(2)[2^3+h(2)]+g(2)[3(2)^2+h'(2)]=11(8+(-6))+5(12+7)=22+95=117$

10. $f'(x)=\dfrac{g(x)(3x^2-2)-g'(x)(x^3-2x)}{(g(x))^2}$
    $f(2)=\dfrac{2^3-2(2)}{g(2)}=\dfrac{4}{8}=\dfrac{1}{2}$
    $f'(2)=\dfrac{g(2)[3(2)^2-2]-g'(2)[2^3-2(2)]}{(g(2))^2}=\dfrac{8(10)-6(4)}{8^2}=\dfrac{56}{64}=\dfrac{7}{8}$
    $f'(5)=\dfrac{g(5)[3(5)^2-2]-g'(5)[5^3-2(5)]}{g(5)^2}=\dfrac{10(73)-15(115)}{10^2}=-\dfrac{995}{100}=-\dfrac{199}{20}$

11. (a) $\dfrac{dy}{dx}=(4\sec^2(4x))(\sin 3x+\sec 5x)+(\tan 4x)(3\cos 3x+5\sec 5x\tan 5x)$

    (b) $\dfrac{dy}{dx}=\dfrac{(1+\cos(4x))(1+5\cos(5x))-(x+\sin(5x))(-4\sin(4x))}{(1+\cos(4x))^2}$

12. $f'(x)=\dfrac{3}{2}(5+4\sqrt{x})^{1/2}\left(\dfrac{4}{2\sqrt{x}}\right)=\dfrac{3\sqrt{5+4\sqrt{x}}}{\sqrt{x}}$
    $g'(x)=4(\cos 3x+\sin 5x)^3(-3\sin 3x+5\cos 5x)$

13. $f'(x) = 6x\sec^2(3x^2)$; $g'(x) = \dfrac{5\cos 5x}{2\sqrt{1+\sin 5x}}$

14. $f'(x) = g(x^2) + 2x^2 g'(x^2)$; $f'(3) = g(9) + 2(3)^2 g'(9) = 10 + 18(-6) = -98$

15. $f'(x) = 3x^2 + 10xg'(5x^2+2)$; $f'(2) = 12 + 20g'(22) = 12 + 20(9) = 192$

16. $3y^2\dfrac{dy}{dx} + 6xy^2 + 3x^2 \cdot 2y\dfrac{dy}{dx} = 8x$; $(3y^2 + 6x^2y)\dfrac{dy}{dx} = 8x - 6xy^2$; $\dfrac{dy}{dx} = \dfrac{8x - 6xy^2}{3y^2 + 6x^2y}$
$\left.\dfrac{dy}{dx}\right|_{(1,2)} = \dfrac{8-24}{12+12} = \dfrac{-16}{24} = -\dfrac{2}{3}$

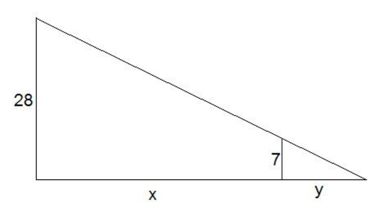
17. (a) Two functions, $f(x) = (4 - x^{2/3})^{3/2}$ and $g(x) = -(4 - x^{2/3})^{3/2}$. Their graphs form the top and bottom of the astroid.
    (b) Tangent line at $(1, 3^{3/2})$: $y = f(1) + f'(1)(x-1)$; $y = 3^{3/2} - 3^{1/2}(x-1)$
    Tangent line at $(1, -3^{3/2})$: $y = g(1) + g'(1)(x-1)$; $y = -3^{3/2} + 3^{1/2}(x-1)$

18. $f'(x) = 3x^2\ln(x^2+4) + \dfrac{2x^4}{x^2+4}$

19. $s^2 = x^2 + y^2$; $2s\dfrac{ds}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}$; $\dfrac{ds}{dt} = \dfrac{1}{s}\left(x\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right)$
When $t = 3$, we have $x = 150$, $y = 210$, $dx/dt = 50$, $dy/dt = 70$, and $s = \sqrt{150^2 + 210^2} = 30\sqrt{74}$.
So $ds/dt = \dfrac{1}{30\sqrt{74}}(150 \cdot 50 + 210 \cdot 70) = 10\sqrt{74}$ mph.

20. $x^2 + y^2 = 100^2$; $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$
Substitute $x = 60$, $y = 80$, $\dfrac{dx}{dt} = 5$. This gives $600 + 160\dfrac{dy}{dt} = 0$, hence $\dfrac{dy}{dt} = -\dfrac{600}{160} = -\dfrac{15}{4}$ ft/sec.

21. $y/7 = (x+y)/28$; $y = x/3$; $dy/dt = (1/3)(dx/dt) = 5/3$ ft/sec.



22. (a) $dy = f'(x)\,dx = \frac{3}{2}x^{1/2}\,dx = \frac{3}{2}(25)^{1/2}\,(2.04) = 15.300$
    (b) $\Delta y = (25 + 2.04)^{3/2} - 25^{3/2} = 15.608$

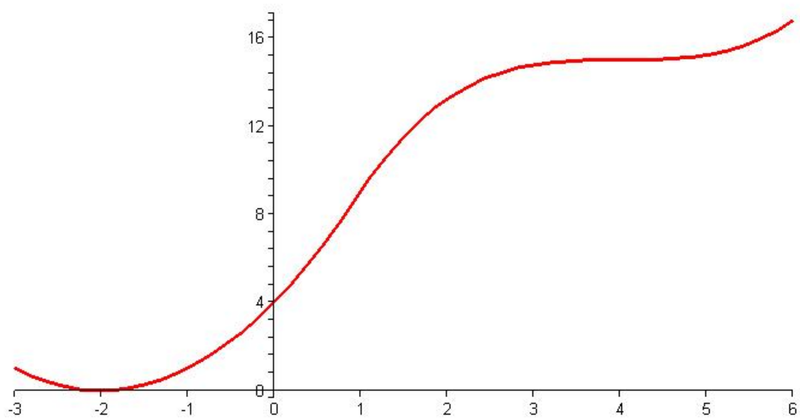
23. $f'(x) = \frac{3}{2}x^{1/2} + \frac{1}{2}x^{-1/2}$; $f(100) = 1010$; $f'(100) = 15.05$
$L(x) = f(100) + f'(100)(x - 100) = 1010 + 15.05(x - 100)$
$L(98.4) = 1010 + 15.05(98.4 - 100) = 985.92$

24. $h'(x) = e^{-2x} - 2xe^{-2x} = e^{-2x}(1-2x)$. The critical point $h'(x) = 0$ occurs at $x = 1/2$. We compute $h(0) = 0$, $h(1/2) = 0.184$, and $h(3) = 0.007$, so the absolute minimum is 0 and the absolute maximum is 0.184.

25. $f'(x) = 36 + 12x - 3x^2 = -3(x^2 - 4x - 12) = -3(x-6)(x+2)$. The critical points are $x = -2$ and $x = 6$, but $x = 6$ is outside the interval. We calculate $f(-3) = 13$, $f(-2) = 0$, and $f(4) = 216$, so the absolute maximum is 216 and the absolute minimum is 0.

26. $\dfrac{f(4) - f(1)}{4 - 1} = \dfrac{176 - 14}{3} = 54$; $f'(x) = 3x^2 + 10x + 8$. The equation to be solved is $3c^2 + 10c + 8 = 54$. The roots of this equation are 2.589 and -5.922, but the latter is outside the interval $[1, 4]$. Therefore $c = 2.589$.

27.

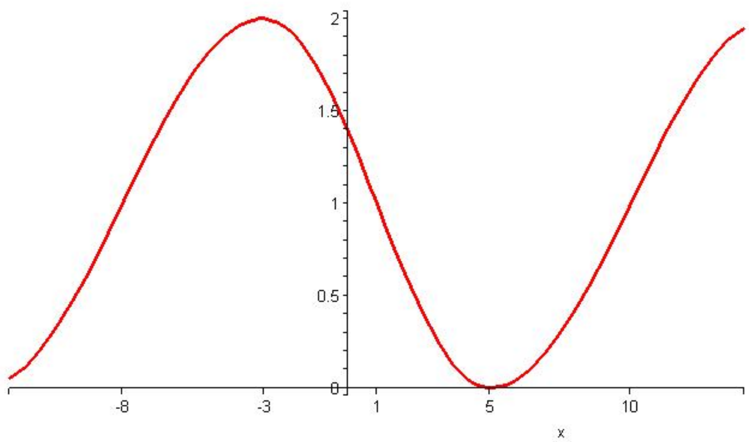


28. $f'(x) = 60 + 18x - 6x^2 = -6(x^2 - 3x - 10) = -6(x-5)(x+2)$. The critical points are $x = 5$ and $x = 2$. Since $f'(0) = 60 > 0$, the derivative is positive on $(2, 5)$, and negative on $(-\infty, 2) \cup (5, \infty)$. Thus f is decreasing on $(-\infty, 2)$, increasing on $(2, 5)$, and decreasing again on $(2, \infty)$, which implies that the local maximum is at $x = 5$ and the local minimum is at $x = 2$.

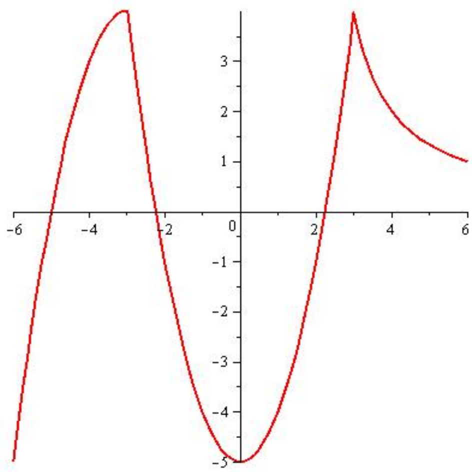
29. $f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6) = 6(x-3)(x+2)$ and $f''(x) = 12x - 6$. The critical points are $x = 3$ and $x = -2$, where the first derivative is 0. Now $f''(3) = 30 > 0$ and $f''(-2) = -30 < 0$, so there is a local minimum at $x = 3$ and a local maximum at $x = -2$.

30. $f'(x) = e^{-x/5} - (x/5)e^{-x/5} = (1 - x/5)e^{-x/5}$
$f''(x) = (-1/5)e^{-x/5} + (1 - x/5)(-1/5)e^{-x/5} = (x/25 - 2/5)e^{-x/5}$
The critical point is $x = 5$ (where the first derivative is 0) and the inflection point is $x = 10$ (where the second derivative is 0). The function is decreasing when $x > 5$ and it is concave downward when $x < 10$.

31.



32.



33. f has local maxima at the critical points $x = 0$ and $x = 10$, because the second derivative is negative at these points. f has a local minimum at the critical point $x = 5$, because $f''(5) < 0$.

34. There are local maxima at $x = -5$ and $x = 8$, because f' changes signs from positive to negative at these points. There is a local minimum at $x = 3$, because f' changes signs from negative to positive at $x = 3$.

35. The critical points are $x = -4$, $x = 1$, and $x = 6$. The critical points $x = 4$ and $x = 6$ give local maxima, because the first derivative changes signs from positive to negative at these points, and also because the second derivative is negative at these points. The critical point $x = 6$ yields a local minimum, because the first derivative changes signs from negative to positive at this point, and also because the second derivative is positive at this point.

36. $\lim_{x\to 0} \dfrac{x \sin 3x}{1 - \cos 4x} = \lim_{x\to 0} \dfrac{\sin 3x + 3x \cos 3x}{4 \sin 4x} = \lim_{x\to 0} \dfrac{6 \cos 3x - 9x \sin 3x}{16 \cos 4x} = \dfrac{6 - 0}{16} = \dfrac{3}{8}$

37. $A = xy$ and $5x + 2y = 1200$. $A = x(1200 - 5x)/2 = 600x - 5x^2/2$. $dA/dx = 600 - 5x$.
$dA/dx = 0$ when $x = 120$ and $y = 300$, so the maximum area is $A = (120)(300) = 36000$ square feet.

38. $V = \pi r^2 h = 288$; $h = 288/r^2$; $C = 16\pi r^2 + 12(2\pi r h) = 16\pi r^2 + 24\pi r(288/r^2) = 16\pi r^2 + 6912\pi/r$; $dC/dt = 32\pi r^2 - 6912\pi/r = 0$; $r^3 = 6912/32 = 216$; $r = 6$; $h = 288/r^2 = 8$. So the cheapest can has radius 6 feet and height 8 feet.

39. Let x be the width and y the height. The problem is to maximize $A = xy$ given that $(x-8)(y-12) = 294$. Solving this constraint for y gives $y = 12 + 294/(x-8)$.
So $A = 12x + 294x/(x-8) = 12x + 294 + 2352/(x-8)$, and $dA/dx = 12 - 2352/(x-8)^2$. This is equal to zero when $(x-8)^2 = 2352/12 = 196$, i.e. when $x = 22$ and $y = 33$.

40. $V = 405 = x^2 y$, so $y = 405/x^2$. The total cost is $C = 18x^2 + 20xy = 18x^2 + 20x(405/x^2) = 18x^2 + 8100/x$. Now $dC/dx = 36x - 8100/x^2 = 0$ when $x = (8100/36)^{1/3} = 15^{2/3}$ and $y = 405/15^{2/3} = \frac{9}{5}15^{2/3}$.

41. (a) Let $f(x) = x^3 - 5x^2 + 7x - 8$. Then $f(3) = 27 - 45 + 21 - 8 = -5 < 0$ and $f(4) = 64 - 80 + 28 - 8 = 4 > 0$, so f has a root in $[3,4]$ by the Intermediate Value Theorem.
    (b) $f'(x) = 3x^2 - 10x + 7$

$$\begin{aligned} x_1 &= 3.5 \\ x_2 &= x_1 - f(x_1)/f'(x_1) = 3.714285714 \\ x_3 &= x_2 - f(x_2)/f'(x_2) = 3.690951518 \\ x_4 &= x_3 - f(x_3)/f'(x_3) = 3.690647499 \end{aligned}$$

42. By calculating $f(x)$ at a few values of x, we find that f has a root between -3 and -2, a root between 1 and 2, and a root between 4 and 5. Newton's method will locate the three roots if we use -3, 1, and 4 as initial guesses.

x_1	-3.00000000	1.00000000	4.00000000
x_2	-2.74285714	1.23076923	4.64285714
x_3	-2.71610729	1.23172634	4.49422319
x_4	-2.71582533	1.23172639	4.48414434
x_5	-2.71582530	1.23172639	4.48409891

43. The subintervals are $[1, 1.5]$, $[1.5, 2]$, $[2, 2.5]$, $[2.5, 3]$, $[3, 3.5]$, and $[3.5, 4]$, so the midpoints are

$$\{1.25, 1.75, 2.25, 2.75, 3.25, 3.75\}$$

and $\Delta x = 0.5$. So the Riemann sum is

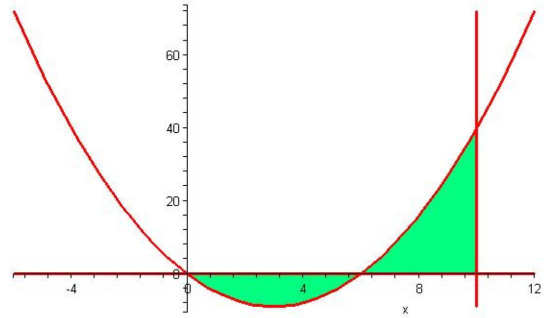
$$[f(x_1) + \ldots + f(x_6)]\Delta x = (f(1.25) + \ldots + f(3.75))(0.5) = 0.6922843210$$

44. The region under the graph of $y = 2 + \sqrt{9 - x^2}$, $0 \le x \le 3$ is a quarter-circle of radius 3, atop a rectangle of height 2 and width 3, so the area under the curve is $(1/4)\pi(3)^2 + 2 \cdot 3$.

45. $f'(x) = \sqrt{2x^3 + 4}$.
Let $G(x) = \int_4^x (t^2 + 8t)^{3/2}\, dt$. Then $g(x) = G(x^2)$, so $g'(x) = 2xG'(x^3) = 2x(x^4 + 8x^2)^{3/2}$.

46. Note that $f(x) \le 0$ when $2 \le x \le 8$. So the area is

$$\int_2^8 -(x^2 - 8x)\, dx = -\frac{x^3}{3} + 4x^2\Big]_2^8 = -\frac{8^3}{3} + 4(8)^2 + \frac{2^3}{3} - 4(2)^2 = 72$$



47. $A = \int_0^6 -(x^2 - 6x)\,dx + \int_6^{10}(x^2 - 6x)\,dx = \dfrac{316}{3}$

48. $du = 2\,dx$; $x = (u-1)/2$.
$\displaystyle\int \frac{x}{\sqrt[3]{2x+1}}\,dx = \int \frac{(u-1)/2}{u^{1/3}}\,\frac{du}{2} = \frac{1}{4}\int u^{2/3} - u^{-1/3}\,du =$
$\displaystyle\frac{1}{4}\left(\frac{3}{5}u^{5/3} - \frac{3}{2}u^{2/3}\right) + C = \frac{3}{20}(2x+1)^{5/3} - \frac{3}{8}(2x+1)^{2/3} + C$

49. $9x^2 = (3x)^2 = (4u)^2 = 16u^2$; $dx = 4/3\,du$

$$\int \frac{dx}{16+9x^2} = \int \frac{4/3du}{16+16u^2} = \frac{64}{3}\frac{du}{1+u^2} = \frac{64}{3}\arctan(u) + C = \frac{64}{3}\arctan(3x/4)$$

50. $u = x^2 + 4$; $du = 2x\,dx$; $du/2 = xdx$; $u(3) = 13$; $u(0) = 4$
$\int_0^3 (x^2+4)^{3/2}\,dx = \int_4^{13} u^{3/2}\tfrac{1}{2}\,du = \tfrac{1}{5}\,u^{5/2}\big]_4^{13} = \tfrac{1}{5}\left(13^{5/2} - 4^{5/2}\right)$

51. $\displaystyle\int_0^3 \frac{x^2}{4+x^2}\,(x\,dx) = \int_4^{13}\frac{u-4}{u}\,\frac{du}{2} = \frac{1}{2}\int_4^{13}\left(1 - \frac{4}{u}\right)\,du =$
$\displaystyle\frac{1}{2}\left[u - 4\ln(u)\right]_4^{13} = \frac{1}{2}[(13 - 4\ln 13) - (4 - 4\ln 4)] = \frac{1}{2}(9 - 4\ln(13/4))$

52. $x = (4/3)\sin\theta$; $dx = 4/3\cos\theta\,d\theta$; $\sqrt{16-9x^2} = \sqrt{16-(3x)^2} = \sqrt{16 - 16\sin^2\theta} = \sqrt{16\cos^2\theta} = 4\cos\theta$.
If $x = 2/3$ then $\theta = \pi/6$, and if $x = 2\sqrt{2}/3$ then $\theta = \pi/4$. So the new integral is

$$\int_{\pi/6}^{\pi/4} \frac{((4/3)\sin\theta)^3}{4\cos\theta}\cdot\frac{4}{3}\cos\theta\,d\theta = \frac{64}{81}\int_{\pi/6}^{\pi/4}\sin^3\theta\,d\theta$$

53. $y = x^2$ and $y = x + 2$ intersect at $x = -1$ and $x = 2$. Since $x + 2 > x^2$ on the interval $[-1, 2]$, the area is $\int_{-1}^2 x + 2 - x^2\,dx = 9/2$.

54. $y = 4x - x^2$ and $y = -2x$ intersect at $x = 0$ and $x = 6$. Since $4x - x^2 > -2x$ on the interval $[0, 6]$, the area is $\int_0^6 4x - x^2 - (-2x)\,dx = 36$.

55.
$$\pi\int_0^{10}[f(x)]^2\,dx = \int_0^{10}\pi(10x - x^2)\,dx = 5x^2 - \frac{1}{3}x^3\Big]_0^{10} = 10000\pi/3$$

56.
$$2\pi\int_0^{10} xf(x) = 2\pi\int_0^{10} x(10x - x^2)\,dx = 5000\pi/3$$

57.

$$\pi \int_0^9 \left([f(x)]^2 - [g(x)]^2\right)\, dx = \pi \int_0^9 9x - x^2\, dx = 243/2$$

58.

$$2\pi \int_0^4 x(f(x) - g(x))\, dx = 2\pi \int_0^4 x(6x - x^2 - 2x)\, dx = 128\pi/3$$

59.

$$\pi \int_0^6 (2 + 6x - x^2)^2 - 2^2\, dx = 2016\pi/5$$

60.

$$2\pi \int_0^6 (x + 3)(6x - x^2)\, dx = 432\pi$$

Solutions

Revised 5 Dec 2007

1. (a) $\dfrac{s(5)-s(0)}{5-0}=2; \quad \dfrac{s(6)-s(2)}{6-2}=5$

   (b) $s'(2)=1$; $s'(5)=7$.

2. (a) $\lim_{x\to 3}\dfrac{x^2+2x-15}{x-3}=\lim_{x\to 3}\dfrac{(x-3)(x+5)}{x-3}=\lim_{x\to 3}x+5=8$

   (b) $\lim_{x\to 2}\dfrac{(x+5)(x^2+2x-8)}{x-2}=\lim_{x\to 2}\dfrac{(x+5)(x-2)(x+4)}{x-2}=\lim_{x\to 2}(x+5)(x+4)=42$

3. $\dfrac{f(x)-f(3)}{x-3}=\dfrac{x^3+8x^2+11x-132}{x-3}=x^2+11x+44$; $f'(3)=3^2+3(11)+44=86.$

4. $\dfrac{f(x)-f(2)}{x-2}=\dfrac{x/(x+5)-2/7}{x-2}=\dfrac{7x-2(x+5)}{7(x+5)(x-2)}=\dfrac{5x-10}{7(x+5)(x-2)}=\dfrac{5}{7(x+5)}$; $f'(2)=\dfrac{5}{49}.$

5. (a) $\lim_{x\to\infty}\dfrac{(2x-5)(3x+1)}{(4x+3)(6x+5)}=\lim_{x\to\infty}\dfrac{(2x-5)(3x+1)/x^2}{(4x+3)(6x+5)/x^2}=\lim_{x\to\infty}\dfrac{(2-5/x)(3+1/x)}{(4+3/x)(6+5/x)}=\dfrac{2\cdot 3}{4\cdot 6}=\dfrac{1}{4}.$

   (b) $\lim_{x\to\infty}\dfrac{(4-3x)(8+5x)}{(2+x)(7+4x)}=\lim_{x\to\infty}\dfrac{(4-3x)(8+5x)/x^2}{(2+x)(7+4x)/x^2}=\lim_{x\to\infty}\dfrac{(4/x-3)(8/x+5)}{(2/x+1)(7+4x)}=\dfrac{(-3)(5)}{(1)(4)}=-\dfrac{15}{4}$

6. $\lim_{x\to\infty}\dfrac{(5+7x)(10+3x)}{(5-2x)(9+8x)}=\lim_{x\to\infty}\dfrac{(5+7x)(10+3x)/x^2}{(5-2x)(9+8x)/x^2}=\lim_{x\to\infty}\dfrac{(5/x+7)(10/x+3)}{(5/x-2)(9/x+8)}=\dfrac{7\cdot 3}{-2\cdot 8}=-\dfrac{21}{16}$

7. $\dfrac{V(8)-V(5)}{8-5}=\dfrac{(4/3)\pi 8^3-(4/3)\pi 5^3}{3}=172\pi$
   $V'(r)=4\pi r^2$, $V'(8)=4\pi 8^2=256\pi$

8. $y'=\dfrac{dy}{dx}=\dfrac{7}{2\sqrt{7x+1}}$; $m=y'(5)=\dfrac{7}{2\sqrt{36}}=\dfrac{7}{12}$; $y=y(5)+m(x-5)$; $y=6+\dfrac{7}{12}(x-5)$; $y=\dfrac{7}{12}x+\dfrac{37}{12}$

9. $f(2)=g(2)(2^3+h(2))=5(8+(-6))=10$
   $f'(x)=g'(x)(x^3+h(x))+g(x)(3x^2+h'(x))$
   $f'(-1)=g'(-1)[(-1)^3+h(-1)]+g(-1)[3(-1)^2+h'(-1)]=4(-1+7)+-3(3+2)=24-15=9$
   $f'(2)=g'(2)[2^3+h(2)]+g(2)[3(2)^2+h'(2)]=11(8+(-6))+5(12+7)=22+95=117$

10. $f'(x)=\dfrac{g(x)(3x^2-2)-g'(x)(x^3-2x)}{(g(x))^2}$

    $f(2)=\dfrac{2^3-2(2)}{g(2)}=\dfrac{4}{8}=\dfrac{1}{2}$

    $f'(2)=\dfrac{g(2)[3(2)^2-2]-g'(2)[2^3-2(2)]}{(g(2))^2}=\dfrac{8(10)-6(4)}{8^2}=\dfrac{56}{64}=\dfrac{7}{8}$

    $f'(5)=\dfrac{g(5)[3(5)^2-2]-g'(5)[5^3-2(5)]}{g(5)^2}=\dfrac{10(73)-15(115)}{10^2}=-\dfrac{995}{100}=-\dfrac{199}{20}$

11. (a) $\dfrac{dy}{dx}=(4\sec^2(4x))(\sin 3x+\sec 5x)+(\tan 4x)(3\cos 3x+5\sec 5x\tan 5x)$

    (b) $\dfrac{dy}{dx}=\dfrac{(1+\cos(4x))(1+5\cos(5x))-(x+\sin(5x))(-4\sin(4x))}{(1+\cos(4x))^2}$

12. $f'(x) = \frac{3}{2}(5+4\sqrt{x})^{1/2}\left(\frac{4}{2\sqrt{x}}\right) = \frac{3\sqrt{5+4\sqrt{x}}}{\sqrt{x}}$
$g'(x) = 4(\cos 3x + \sin 5x)^3(-3\sin 3x + 5\cos 5x)$

13. $f'(x) = 6x\sec^2(3x^2)$; $g'(x) = \dfrac{5\cos 5x}{2\sqrt{1+\sin 5x}}$

14. $f'(x) = g(x^2) + 2x^2 g'(x^2)$; $f'(3) = g(9) + 2(3)^2 g'(9) = 10 + 18(-6) = -98$

15. $f'(x) = 3x^2 + 10xg'(5x^2+2)$; $f'(2) = 12 + 20g'(22) = 12 + 20(9) = 192$

16. $3y^2\dfrac{dy}{dx} + 6xy^2 + 3x^2 \cdot 2y\dfrac{dy}{dx} = 8x$; $(3y^2 + 6x^2y)\dfrac{dy}{dx} = 8x - 6xy^2$; $\dfrac{dy}{dx} = \dfrac{8x - 6xy^2}{3y^2 + 6x^2y}$
$\left.\dfrac{dy}{dx}\right|_{(1,2)} = \dfrac{8-24}{12+12} = \dfrac{-16}{24} = -\dfrac{2}{3}$

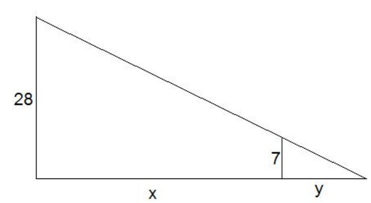
17. (a) Two functions, $f(x) = (4 - x^{2/3})^{3/2}$ and $g(x) = -(4 - x^{2/3})^{3/2}$. Their graphs form the top and bottom of the astroid.
(b) Tangent line at $(1, 3^{3/2})$: $y = f(1) + f'(1)(x-1)$; $y = 3^{3/2} - 3^{1/2}(x-1)$
Tangent line at $(1, -3^{3/2})$: $y = g(1) + g'(1)(x-1)$; $y = -3^{3/2} + 3^{1/2}(x-1)$

18. $f'(x) = 3x^2\ln(x^2+4) + \dfrac{2x^4}{x^2+4}$

19. $s^2 = x^2 + y^2$; $2s\dfrac{ds}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}$; $\dfrac{ds}{dt} = \dfrac{1}{s}\left(x\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right)$
When $t = 3$, we have $x = 150$, $y = 210$, $dx/dt = 50$, $dy/dt = 70$, and $s = \sqrt{150^2 + 210^2} = 30\sqrt{74}$.
So $ds/dt = \dfrac{1}{30\sqrt{74}}(150 \cdot 50 + 210 \cdot 70) = 10\sqrt{74}$ mph.

20. $x^2 + y^2 = 100^2$; $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$
Substitute $x = 60$, $y = 80$, $\dfrac{dx}{dt} = 5$. This gives $600 + 160\dfrac{dy}{dt} = 0$, hence $\dfrac{dy}{dt} = -\dfrac{600}{160} = -\dfrac{15}{4}$ ft/sec.

21. $y/7 = (x+y)/28$; $y = x/3$; $dy/dt = (1/3)(dx/dt) = 5/3$ ft/sec.



22. (a) $dy = f'(x)\,dx = \frac{3}{2}x^{1/2}\,dx = \frac{3}{2}(25)^{1/2}\,(2.04) = 15.300$
(b) $\Delta y = (25 + 2.04)^{3/2} - 25^{3/2} = 15.608$

23. $f'(x) = \frac{3}{2}x^{1/2} + \frac{1}{2}x^{-1/2}$; $f(100) = 1010$; $f'(100) = 15.05$
$L(x) = f(100) + f'(100)(x - 100) = 1010 + 15.05(x - 100)$
$L(98.4) = 1010 + 15.05(98.4 - 100) = 985.92$