

Lecture 4: Business Applications of Algebra

In this lecture, we will study the following topics:

1. Function notation
2. Piecewise-defined functions
3. Depreciation
4. Break-even analysis
5. Supply and demand

Function notation

We say that f is a function of x if the value of f is uniquely determined by the value of x . We often write $f(x)$ to indicate that f is a function of x . We say that x is the independent variable, while f is the dependent variable.

Functions are often defined by formulas. We **evaluate** $f(x)$ by replacing x with a number. For example, suppose that f is defined by the formula $f(x) = 2x + 3$. To evaluate $f(x)$ at $x = 5$, we calculate $f(5) = 2(5) + 3 = 13$.

Sending a package by first-class mail costs 39 cents for the first ounce, and 24 cents for each additional ounce, up to 13 ounces. Thus, the cost C of sending a package by first-class mail is a function of the weight w of the package, and we can describe this function by the formula $C(w) = 0.24w + 0.15$. To calculate the cost of mailing a six ounce package, we replace w with 6.

$$C(6) = 0.24(6) + 0.15 = 1.44 + 0.15 = \$1.59.$$

Exercise: Let $f(x) = 10x + 50$. Calculate $f(10)$ and $f(-10)$. (Answers: $f(10) = 600$, $f(-10) = 400$.)

Piecewise-defined functions

A function is defined **piecewise** if it is given by different expressions on various intervals. For example, let

$$f(x) = \begin{cases} x+2 & \text{if } x \leq 0 \\ 3x & \text{if } 0 < x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$$

In order to evaluate f , we need to determine which expression applies. If $x \leq 0$ then the first case applies; for example $f(-3) = (-3) + 2 = -1$. If $0 < x < 2$ then the second case applies; for example $f(1.2) = 3(1.2) = 3.6$. Finally, if $x \geq 2$ then the third case applies; for example $f(2) = 2^2 = 4$.

Exercise: Let $g(x) = \begin{cases} x & \text{if } 0 \leq x \leq 10; \\ 2x & \text{if } 10 < x \leq 20; \\ 3x+1 & \text{if } x > 20. \end{cases}$ Find $g(5)$, $g(10)$, $g(20)$, and $g(30)$.

Answers: $g(5) = 5$, $g(10) = 10$, $g(20) = 40$, $g(30) = 61$.

Piecewise-defined functions are often used for income taxation. Suppose that a country has the following income tax structure. The first \$15,000 of income is not taxed. Income between \$15,000 and \$40,000 is taxed at a marginal rate of 20%. Income between \$40,000 and \$80,000 is taxed at a marginal rate of 35%. Finally, all income exceeding \$80,000 is taxed at a marginal rate of 50%.

Now, a person who earned exactly \$15,000 would pay no income tax. If he earned \$40,000, then he would pay no tax on the first \$15,000, but he would have to pay 20% on the remaining \$25,000, so he would pay an income tax of \$5,000. If he earned exactly \$80,000, then he would pay a tax of \$5,000 on his first \$40,000, and he would pay 35% on the remaining \$40,000, so his income tax would be $\$5,000 + .35(\$40,000) = \$19,000$.

We can use this data to write a formula for $T(x)$, the amount of taxes owed on an income of x . The function is defined piecewise. Each piece is a linear function whose slope is equal to the marginal tax rate. Using the point-slope formula from the previous lecture, we obtain the following.

$$T(x) = \begin{cases} 0 & \text{if } x \leq 15000, \\ 0.2(x - 15000) & \text{if } 15000 < x \leq 40000, \\ 5000 + 0.35(x - 40000) & \text{if } 40000 < x \leq 80000, \\ 19000 + 0.5(x - 80000) & \text{if } 80000 < x. \end{cases}$$

Exercise: A new income tax structure has been proposed. There will be no taxes on the first \$20,000 of income, but income between \$20,000 and \$75,000 will be taxed at a marginal rate of 25%, and income exceeding \$75,000 will be taxed at a marginal rate of 45%. Write a formula for this tax structure.

Answer:

$$T(x) = \begin{cases} 0 & \text{if } x \leq 20000, \\ 0.25x - 5000 & \text{if } 20000 < x \leq 75000, \\ 0.45x - 20000 & \text{if } 75000 < x. \end{cases}$$

Depreciation

Many businesses must invest in machinery or equipment that is required in order for the business to be successful. Some items will be used up within a year – those items are deducted from revenue in the year they are purchased. Some other items, such as major pieces of machinery, are deducted over several years. This period of time is determined by the estimated number of years that the item is expected to be useful and is called the **expected life** or the **useful life** of the item.

The value of an item decreases each year until the end of its useful life. The amount by which the equipment decreases in value per year is called the **annual depreciation**. We will assume that the item depreciates by the same amount every year throughout its useful lifetime – this is called **straight-line** or **linear** depreciation. At the end of its useful years, an item has a **salvage value**. Other terms are **residual value**, **scrap value**, or **trade-in value**. We assume that this value stays the same forever after the useful life has expired.

When using straight-line depreciation, we can calculate the annual depreciation using the following formula:

$$\text{Annual depreciation} = \frac{(\text{Original cost}) - (\text{Salvage value})}{\text{Expected life}} \quad \text{or} \quad AD = \frac{OC - SV}{L}.$$

The **book value** after x years (during the useful lifetime) is given by the following formula:

$$BV = OC - AD \cdot x$$

Example: A piece of equipment has an original cost of \$5,000, a salvage value of \$750, and a useful life of 5 years. Find the annual depreciation and the book value at the end of each year.

Solution: We have $OC = 5000$, $SV = 750$, and $L = 5$. The annual depreciation is

$$AD = \frac{OC - SV}{L} = \frac{5000 - 750}{5} = \$850$$

A **depreciation schedule**, showing the accumulated depreciation and the book value at the end of each year, would look like this:

Year	Original cost	Annual depreciation	Accumulated depreciation	Book value
1	\$5,000	\$850	\$850	\$4,150
2	\$5,000	\$850	\$1,700	\$3,300
3	\$5,000	\$850	\$2,550	\$2,600
4	\$5,000	\$850	\$3,400	\$1,300
5	\$5,000	\$850	\$4,150	\$750

We can also describe straight-line depreciation with a piecewise-defined formula:

$$BV(x) = \begin{cases} OC - AD \cdot x & \text{if } 0 \leq x < L, \\ SV & \text{if } L \leq x. \end{cases}$$

For example, suppose that a new computer costs \$2000. Owing to technological progress, it is anticipated that the computer will have a useful life of 5 years, after which it will have a salvage value of \$200. Then the computer depreciates by $(\$2000 - \$200)/5 = \$360$ per year, and the formula for the book value is

$$BV(x) = \begin{cases} 2000 - 360x & \text{if } 0 \leq x < 5, \\ 200 & \text{if } L \leq x. \end{cases}$$

Exercise: A company purchases a hydraulic press for \$5,000. The machine has a useful life of 15 years, and a salvage value of \$500. Write a formula for the book value of the press, assuming linear depreciation.

Answer:

$$BV(x) = \begin{cases} 5000 - 300x & \text{if } 0 \leq x < 15, \\ 500 & \text{if } L \leq x. \end{cases}$$

Break-even analysis

The break-even point for a product is the point where total revenue TR equals the total cost TC associated with sales of the product. In order to make a profit, the firm must ensure that $TR > TC$. The total profit is equal to total revenue minus total cost ($TR - TC$). Revenue is equal to the sale price P times the quantity sold Q . ($TR = P \times Q$)

The total cost C has two components. There is a fixed cost FC that is incurred regardless of the number of items produced, and there is a variable cost VC that increases with quantity. The variable cost is equal to the marginal cost times the quantity ($VC = MC \times Q$). We assume here that the marginal cost (i.e. the cost to produce a single item) is always the same regardless of the quantity produced, although this is not always true in real life.

Exercise: Joe plans to sell lemonade at the county fair for 75 cents per cup. It costs \$40 to rent a concession stand for a day, but there are no other fixed costs. Joe expects that each cup of lemonade will cost him 25 cents for materials. How many cups of lemonade must be sell to make a profit?

Solution: The total revenue is $TR = P \times Q = 0.75 Q$, where Q is the number of cups sold. The fixed cost is $FC = \$40$, and the marginal cost is $MC = 0.25$. So the total cost is $TC = 40 + 0.25 Q$. To find the break-even point, we set the total cost equal to the total revenue and solve for Q .

$$0.75Q = 40 + 0.25Q$$

$$0.75Q - 0.25Q = 40$$

$$0.5Q = 40$$

$$Q = 80$$

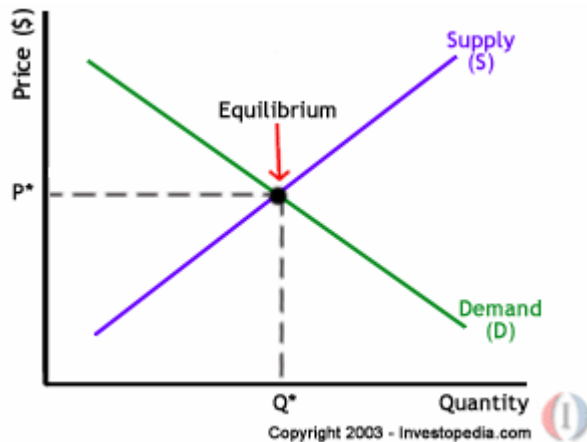
Joe breaks even after selling 80 cups, but he must sell at least one additional cup to turn a profit. Therefore our answer is 81.

Supply and demand

Supply and demand curves are used in economics to explain changes in the price and quantity sold of goods sold in competitive markets. In this model, the vertical axis represents the price of an item, and the horizontal axis represents the quantity sold.

The supply curve indicates the quantity that producers are willing to sell at each possible price. As the price increases, suppliers are willing to produce more items, so the supply curve is upward-sloping. The demand curve indicates the total quantity that consumers are willing to purchase at each possible price. As the price increases, consumers will purchase less of the good, so the demand curve is downward sloping.

The point where the supply and demand curves cross is called the **market equilibrium point**. The market equilibrium point indicates the market price and the number of items to be sold. External factors may cause the supply curve or the demand curve to be shifted, and this will alter the market equilibrium.



The web site <http://www2.yk.psu.edu/~dx131/econ14/notes.html> has an excellent tutorial on supply and demand.