WEEK 3: LINEAR MODELS I

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These lecture notes are incomplete, so you should also read the other set of lecture notes for Week 3, in Document Sharing.

Linear equations in two variables

Our main objective this week is to understand linear equations and systems of linear equations in two variables.

A linear equation in two variables x and y is an equation that can be put in the form $Ax + By = C$, where A, B, and C are real numbers. A solution to such an equation is a pair of numbers – an x value and a y value.

For example, $2x + 3y = 11$ is a linear equation. One of the solutions of this equation is $\{x = 4, y = 1\}$, because we get a true statement when we replace x with 4 and y with 1.

$$2(4) + 3(1) = 11$$

Another solution is $\{x = 7, y = -1\}$.

$$2(7) + 3(-1) = 11$$

For brevity, we write these solutions as $(4, 1)$ and $(7, -1)$. This is called **ordered pair notation**. The first number in the ordered pair is the value of x, and the second number is the value of y.

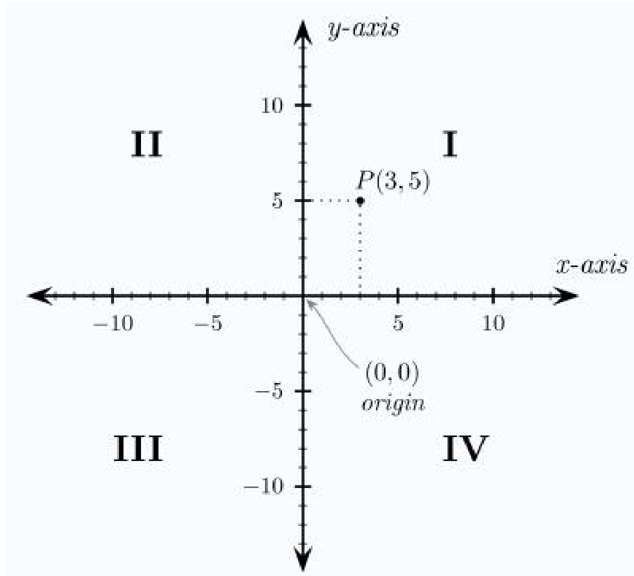
Exercise: Five linear equations and five ordered pairs are given. Match each linear equation with a solution.

	Equation		**Ordered pair**
A.	$y = 3x + 6$	**a.**	$(1, 1)$
B.	$x = 2$	**b.**	$(1, 9)$
C.	$3x + 2y = 5$	**c.**	$(5, 10)$
D.	$y = 15 - x$	**d.**	$(2, 5)$
E.	$x - 4y = 7$	**e.**	$(3, -1)$

Answers: $A \implies b$, $B \implies d$, $C \implies a$, $D \implies c$, $E \implies e$

Cartesian coordinate system

We use the Cartesian coordinate system to graph solutions to equations in two variables. In this system, there are two number lines, one horizontal and one vertical. The horizontal number line is called the x-axis and the vertical number line is called the y-axis. The two number lines meet at a point called the **origin**, which is the zero point of both axes.



The x-axis and the y-axis divide the plane into four regions, called **quadrants**. The quadrants are numbered from I to IV (using Roman numerals), in the counter-clockwise direction, starting with the upper right.

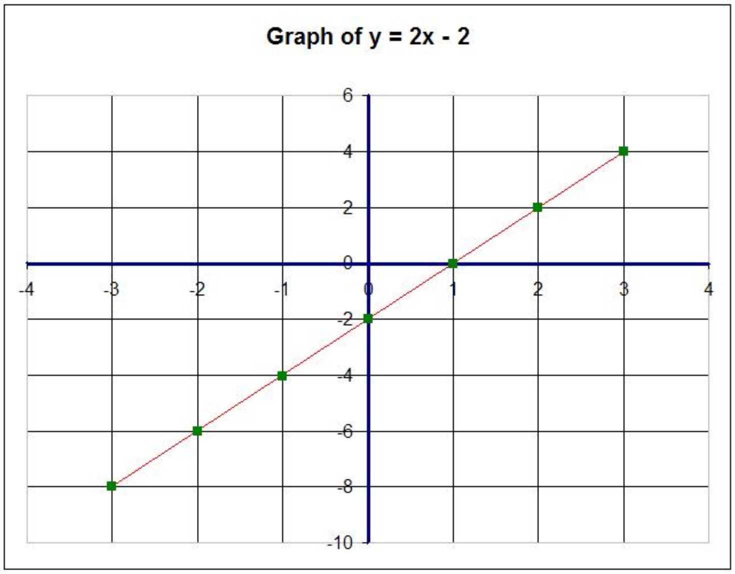
To locate the point (u, v), we start at the origin, then we move u units to the right, and v units up. If u is negative then we move to the left instead of right, and if v is negative then we move down instead of up.

Graphing lines

Let us consider the equation $y = 2x - 2$. We can find solutions to this equation by substituting values for x and calculating the corresponding values of y. We call this a **table of values**.

x	y	point
-3	-8	$(-3, -8)$
-2	-6	$(-2, -6)$
-1	-4	$(-1, -4)$
0	-2	$(0, -2)$
1	0	$(1, 0)$
2	2	$(2, 2)$
3	4	$(3, 4)$

When we plot these points, we observe that they all lie on a straight line. We say that this line is the **graph** of the equation $y = 2x - 2$.



We can use this approach to draw the graph of any linear equation. When using this approach it is helpful to solve the equation for y before making the table of values. For example, in order to graph the equation $3x + 4y = 5$, we would rewrite the equation as $y = -3/4x + 5/4$.

The TI-83 Plus calculator can compute tables of values. Press the Y= button, enter the desired equation, then press 2nd TABLE. The table starts with $x = 0$ by default, but you can change the starting value by pressing 2nd TBLSET.

Intercepts

The x-intercept of a line is the point where the line crosses the x-axis, and the y-intercept of a line is the point where the line crosses the y-axis. To find the x-intercept, we set $y = 0$ and solve for x. To find the y-intercept, we set $x = 0$ and solve for y.

Exercise: Find the x-intercept and the y-intercept of each line.

(1) $y = 3x + 6$
(2) $x + 2y = 12$
(3) $x = 4 + 8y$

Answers: (1) $x = -2$, $y = 6$ (2) $x = 12$, $y = 6$ (3) $x = 4$, $y = -1/2$

Slope of a line

The slope of a line is a number that describes how steep the line is. The slope of a horizontal line is zero. A line has positive slope, or is increasing, if the line goes upward as one moves from left to right along the line. A line has negative slope, or is decreasing, if the line goes downward as one moves from left to right along the line.

A vertical line (straight up and down) is neither increasing nor decreasing, and it does not have a slope. We say that the slope of a vertical line is **undefined**. Note that undefined is not the same thing as zero. A horizontal line has a slope of zero, but a vertical line has no slope, not even zero.

To calculate the slope of a line, we pick any two points (x_1, y_1) and (x_2, y_2) on the line. The slope m is the change in the y-values (the "rise") divided by the change in the x-values (the "run").

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

It turns out that the slope does not depend on which points you pick. No matter which two points you pick from the line, you will always get the same value for the slope.

Exercise: Find the slope of the line passing through $(1, -2)$ and $(3, 6)$.

Solution: $m = \dfrac{6-(-2)}{3-1} = \dfrac{8}{2} = 4.$

Slope-intercept form of a line

The **slope-intercept form** of a line is

$$y = mx + b$$

It is so called because m is the slope of the line, and b is the y-intercept. Writing the equation of a line in slope-intercept form is simply a matter of solving the equation for y. All lines can be expressed in slope-intercept form, except for vertical lines, which have the form $x = a$.

Exercises: (1) Find the slope and the y-intercept of the line $y = -x + 15$. (2) Express $x + 2y = 10$ in slope-intercept form.

Answers: (1) The slope is -1 and the y-intercept is 15. (2) $y = -1/2\ x + 5$.

Point-slope form of a line

The point-slope formula gives the equation of a line, given the slope of the line, and a point on the line. The point-slope form of a line, with slope m, passing through the point (x_1, y_1), is as follows.

$$y - y_1 = m(x - x_1)$$

For example, $y + 4 = 3(x - 2)$ is the equation of the line with slope $m = 3$ passing through the point $(2, -4)$.

Exercises:

(1) Find the equation of the line with slope -4, passing through the point $(1, 2)$.
(2) Find the equation of the line passing through the points $(1, 3)$ and $(2, 5)$.

Solutions:

(1) $y - 2 = -4(x - 1)$.
(2) The slope is $m = (5-3)/(2-1) = 2$, so the equation is $y - 3 = 2(x - 1)$, or $y = 2x + 1$.

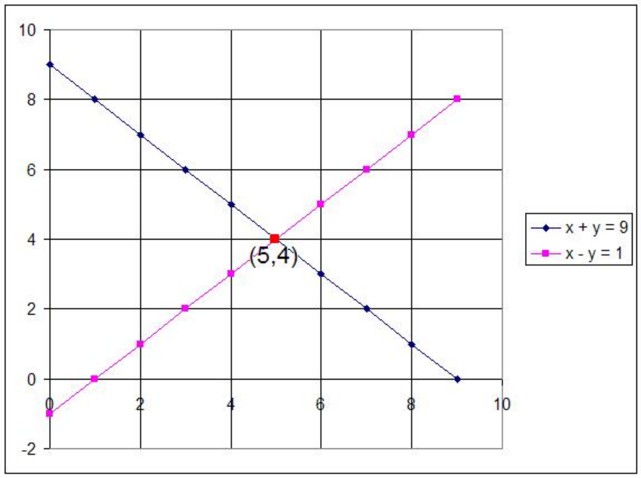
Systems of equations in two variables

A **system of equations** is a collection of two or more equations, considered together. If there is a solution that is common to all equations in the system, then we say that it is a solution of the system. For example, let us consider the following system of equations.

$$\begin{aligned} x + y &= 9 \\ x - y &= 1 \end{aligned}$$

Note that $(3, 6)$ is a solution to the first equation, but not the second; since $3 + 6 = 9$, but $3 - 6 \neq 1$. However, $(5, 4)$ is a solution to both equations, since $5 + 4 = 9$ and $5 - 4 = 1$. We say that $(5, 4)$ is the solution of this system of linear equations.

One way to solve a system of linear equations in two variables is to graph the lines and locate the point where they cross. In our example, the lines $x + y = 9$ and $x - y = 1$ intersect at the point $(5, 4)$.



Substitution method

The graphical method for solving systems of equations is time-consuming and imprecise, so we need alternative methods. One alternative is the **substitution method**, which is described below.

(1) Solve the first equation for y.
(2) Substitute the resulting expression for y in the second equation. This gives an equation in x only.
(3) Solve the equation from step (2) to find the value of x.
(4) Substitute this value of x into the equation from step (1).
(5) Use this to find the value of y.

Let's try this with the system $x + y = 9$, $x - y = 1$. Solving the first equation for y gives $y = 9 - x$. We replace y with $9 - x$ in the second equation, giving $x - (9 - x) = 1$. The solution to that equation is $x = 5$. Finally we replace x with 5 in the equation $y = 9 - x$. This yields $y = 9 - 5 = 4$. So the solution of the system is $(5, 4)$.

Elimination method

In this case, we can find the solution more quickly by adding and subtracting the equations. When we add the equations together, the y's cancel, and we can find x.

$$\begin{aligned} x + y &= 9 \\ x - y &= 1 \\ 2x &= 10 \qquad x = 5 \end{aligned}$$

When we subtract the equations, the x's cancel, and we can find y. Recall that subtracting is the same as adding the negative.

$$\begin{aligned} x + y &= 9 \\ -x + y &= -1 \\ 2y &= 8 \qquad y = 4 \end{aligned}$$

The following system of equations is more typical.

$$\begin{aligned} 2x + 3y &= 8 \\ 3x + 4y &= 11 \end{aligned}$$

Note that the variables will not cancel just by adding or subtracting the equations. But if we multiply each equation by the right number, then we can cleverly arrange that addition will cause a variable to be canceled.

To solve for y, we can multiply the first equation by 4, and the second equation by -3.

$$\begin{aligned} 8x + 12y &= 32 \\ -9x - 12y &= -33 \\ -x &= -1 \\ x &= 1 \end{aligned}$$

(Why 4 and -3? We observed that $12y$ is the least common multiple of $3y$ and $4y$, so we arranged that the first equation would have the term $12y$, and the second equation would have the term $-12y$.)

To solve for x, we multiply the first equation by 3, and the second equation by -2.

$$\begin{aligned} 6x + 9y &= 24 \\ -6x - 8y &= -22 \\ y &= 2 \end{aligned}$$

Matrix Row Reduction

When solving systems of equations with several unknowns, it is convenient to suppress the variable names and the equals signs. Instead, we arrange the coefficients in tabular form, with rows and columns. This arrangement is called a **matrix**. Each row of the matrix represents an equation in the system. Each column represents an unknown, except for the last column.

The process is best understood by example. Consider the following system of equations.

$$\begin{aligned} x + 2y + 3z &= 7 \\ 2x + y - z &= 0 \\ x + 4y + 2z &= 5 \end{aligned}$$

The matrix that represents this system is

$$\begin{bmatrix} 1 & 2 & 3 & 7 \\ 2 & 1 & -1 & 0 \\ 1 & 4 & 2 & 5 \end{bmatrix}$$

We can solve the system by applying the following **elementary row operations**.

(1) Exchange two rows.
(2) Multiply a row by a nonzero number.
(3) Add a multiple of one row to another row.

By applying these row operations, we can transform our matrix to the following matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Notice that the new matrix has 1s along the diagonal, starting from the top left corner, and that all other entries are 0, except in the last column. This is called "reduced row echelon form", or RREF. The TI-83 Plus calculator can compute the RREF – consult your manual for details.