Lecture 2: Introduction to Algebra

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January 12, 2006

1 Algebraic expressions

An algebraic expression may contain *variables* such as x, y, z, etc. Evaluating an expression means that we substitute the given values for the variables, and perform the indicated calculations, following the order of operations. It is advisable to enclose the substituted values with parentheses, especially if they are negative.

Example: Evaluate $b^2 - 4ac$ when $a = 1$, $b = -3$, and $c = -5$.

Solution: $b^2 - 4ac = (-3)^2 - 4(1)(-5) = 9 - (-20) = 9 + 20 = 29$

Two expressions are **equivalent** if they have the same value for all possible values of the variables. For example, $x + x$ and $2x$ are equivalent expressions. To **simplify** an expression is to replace a complicated expression with an equivalent expression that is shorter. It is desirable to simplify expressions as far as possible. We simplify expressions by using the following rules.

$x + y = y + x$	Commutative law of addition
$x + 0 = 0 + x = x$	Additive identity law
$(x + y) + z = x + (y + z)$	Associative law of addition
$x + (-x) = (-x) + x = 0$	Additive inverse law
$x \cdot y = y \cdot x$	Commutative law of multiplication
$x \cdot 1 = 1 \cdot x = x$	Multiplicative identity law
$x \cdot 1/x = 1$, when $x \neq 0$	Multiplicative inverse law
$(x \cdot y) \cdot y = x \cdot (y \cdot z)$	Associative law of multiplication
$(x + y)z = xz + yz$	Distributive law
$x \cdot 0 = 0$	Multiplication by zero

Example: Simplify $5(x + 2) + x$

Solution:

$5(x+2)+x$	$=$	$(5x + 5 \cdot 2) + x$	Distributive law
	$=$	$(5x + 10) + x$	$5 \times 2 = 10$
	$=$	$x + (5x + 10)$	Commutative law of addition
	$=$	$(x + 5x) + 10$	Associative law of addition
	$=$	$(1x + 5x) + 10$	Multiplicative identity law
	$=$	$(1 + 5)x + 10$	Distributive law
	$=$	$6x + 10$	$1 + 5 = 6$

It is tedious to include every single step in such a calculation, so one usually writes something like this:

$$5(x + 2) + x = 5x + 10 + x = 6x + 10$$

The important thing to note in this example is that we can add like terms by adding their coefficients. For example, $5x$ and $8x$ are like terms, and $5 + 8 = 13$, so $5x + 8x = 13x$. The sum of unlike terms, such as $5x + 8y$, cannot be simplified.

Rules for removing parentheses

When an addition or subtraction sign appears immediately after a quantity in parentheses, the parentheses can be removed. When an addition sign appears immediately in front of a quantity in parentheses, the parentheses can be removed. When a subtraction sign appears immediately in front of a quantity in parentheses, we may also remove the parentheses, but we must change the sign of each term inside the parentheses.

Examples:

$$\begin{aligned} (2x-8)+(3y-z+15) &= 2x-8+3y-z+15 \\ x-(2y-3z+24) &= x-2y+3z-24 \end{aligned}$$

Exercises: Simplify each expression

1. $4 - 5x + 6y + 7x + 8$
2. $3(x + 5)$
3. $4 - (x - 5)$
4. $2(x + 1) - (-3)(x - 4) - (5x - 6)$

Answers: $12 + 2x + 6y$, $3x + 15$, $9 - x$, -4

2 Equations

An **equation** is a statement that two things are equal; it consists of two algebraic expressions, with an equals sign between them. A **solution** to an equation is an assignment of values to the variables which makes the equation true. For example, $x = 5$ is a solution to the equation $3x + 1 = 16$, because $3(5) + 1 = 16$ is a true statement.

To solve an equation for a given variable x, we must transform the equation to an equivalent equation so that x is isolated on the left side of the equation and x does not appear on the right side. We can use the following methods to transform an equation.

1. Simplify either or both sides of the equation, using the properties of real numbers from the previous section.
2. Add or subtract the same quantity to both sides of the equation.
3. Multiply or divide both sides of the equation by the same nonzero quantity.
4. Swap the two sides of the equation.

As a shortcut, we can move a term from one side of an equation to the other side by reversing its sign. (A term is something that is added or subtracted.) For example, $4x-7 = 3x+8$ is equivalent to $4x-3x = 8+7$. The -7 changed to 7 when it was moved to the other side of the equation, and $3x$ changed to $-3x$.

An equation in the variable x is **linear** if each side of the equation can be simplified to an expression of the form $Ax + B$, where A and B are real numbers. To solve a linear equation in x, follow these steps.

1. If fractions are present, multiply both sides by the least common denominator (LCD).
2. Use the distributive law to remove all parentheses.
3. Move the variable terms (such as $2x$) to one side of the equation, and constant terms (such as -6) to the other side.
4. Combine like terms on each side of the equation.
5. Divide both sides by the coefficient of the variable.
6. Check the solution by substituting into the original equation.

Example: Solve $4x + 3 = 7x - 14$

Solution

$$\begin{aligned} 4x - 7x &= -14 - 3 \\ -3x &= -17 \\ x &= (-17)/(-3) \\ x &= 17/3 \end{aligned}$$

Example: Solve $xy + yz = 2xz - y^2$ for x

Solution

$$\begin{aligned} xy - 2xz &= -y^2 - yz && \text{Move terms with } x \text{ to the left, other terms to the right} \\ x(y - 2z) &= -y^2 - yz && \text{Distributive law} \\ x &= \frac{-y^2 - yz}{y - 2z} && \text{Divide both sides by the coefficient of } x \end{aligned}$$

The TI-83 Plus has a very powerful equation solver. The solver is accessed by pressing [MATH] [0]. Consult section 2-8 of the manual for detailed instructions.

3 Inequalities

Inequalities differ from equations in that they use an inequality symbol instead of equals sign. The most common inequality symbols are shown in the table below.

Symbol	**Meaning**
$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to
$\neq$	not equal to

The number line helps us to visualize inequalities. The statement $A < B$ means that A lies to the left of B on the number line, and $A > B$ means that A lies to the right of B on the number line. The statement $A \leq B$ means that A is not greater than B.

Let A, B, and C be real numbers. Then the following statements hold.

- If $A < B$ then $B > A$.
- If $A < B$ and $B < C$ then $A < C$.
- Exactly one of the following is true: $A < B$, $A = B$, $A > B$.
- If $A < B$ then $A + C < B + C$.
- If $A < B$ and $C > 0$ then $AC < BC$.
- If $A < B$ and $C < 0$ then $AC > BC$.

You solve a linear inequality in exactly the same way that you solve a linear equation, with only two exceptions:

1. If you multiply or divide both sides by a negative number, you must reverse the inequality.
2. If you exchange the sides of an inequality, you must reverse the inequality (for example, $2 \leq x$ becomes $x \geq 2$).

Note that adding and subtracting will never reverse the direction of an inequality.

Example: Solve $2 - 3x \leq 10 + x$.

Solution:

$$\begin{aligned} 2 - 3x &\leq 10 + x \\ -3x - x &\leq 10 - 2 \\ -4x &\leq 8 \\ x &\geq 8/(-4) \\ x &\geq -2 \end{aligned}$$

4 Word problems

Solving word problems.

1. Read the problem carefully. Determine which information is given to you in the problem, and which information you are asked to find.
2. Choose a variable to represent the unknown quantity. State explicitly what the variable represents (e.g. **Let x = number of dimes**).
3. Translate the problem into an equation. Drawing a picture is often helpful.
4. Solve the equation.
5. Answer the question asked in the problem!
6. Check your solution by using the original words of the problem.

Translating key words and phrases.

Key words and phrases	Verbal description	Algebraic Statement
Equality		
Equals, equals to, is, are, was, will be, represents	The sales price S is \$10 less than the list price L.	$S = L - 10$
Addition		
Sum, plus, greater than,	The sum of 5 and x	$5 + x$
increased by, more than,	Seven more than y	$y + 7$
exceeds, total of	x exceeds y by 2	$x = y + 2$
Subtraction		
Difference, subtracted from	The difference of 4 and b	$4 - b$
less, minus, decreased by,	Three less than z	$z - 3$
reduced by, the remainder	The width reduced by 1	$w - 1$
Multiplication		
Product, multiplied by,	Two times x	$2x$
twice, times, percent of	Five percent of x	$.05x$
Division		
Quotient, divided by,	The quotient of x and 8	$\frac{x}{8}$
ratio, per	The ratio of x to y	x/y

5 Ratio and proportion

A **ratio** is a quotient of two quantities. They are often expressed using the word "per." (Miles per gallon, ten percent, etc.) Ratios may be expressed in many ways, including:

$$a \text{ to } b, \qquad a : b, \qquad a/b, \quad \text{and} \quad \frac{a}{b}.$$

A **proportion** is a statement that two ratios are equal.

Cross multiplication. $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. For example, to solve $\frac{6}{x} = \frac{4}{3}$, we could cross multiply to obtain $6 \cdot 3 = x \cdot 4$.

We can also speak of ratios between three or more quantities. For example, the numbers 30, 60, 90 are in the ratio $1 : 2 : 3$. If three numbers are in the ratio $A : B : C$ then we can express the numbers as Ax, Bx, and Cx for some nonzero number x.

Example: Three brothers, Mark, John and Luke, split an inheritance of \$500,000 in the ratio 5 : 3 : 2. How much does each brother get?

Solution: Mark receives $5x$, John receives $3x$, and Luke receives $2x$. Then $5x + 3x + 2x = 500,000$, so $10x = 500,000$ and $x = 50,000$. Mark's share is $5 \times \$50,000 = \$250,000$, John's share is $3 \times \$50,000 = \$150,000$, and Luke's share is $2 \times \$50,000 = \$100,000$.

6 Geometric formulas

Perimeter

The **perimeter** of a figure is the length of the boundary. For a polygon (such as a triangle or rectangle) the perimeter is just the sum of the lengths of the sides. The perimeter of a circle is called the **circumference**. The circumference C of a circle can be found using the formula

$$C = \pi d = 2\pi r$$

where d is the diameter, and r is the radius. The diameter is the distance across the circle. The radius is the distance from the center of the circle to the boundary, and it is half the diameter. The number π is approximately 3.141592654, but you don't need to remember the digits, because your calculator has a $\boxed{\pi}$ button.

Area

The area of a rectangle is the product of the length and the width ($A = L \times W$). If the rectangle is a square then L is equal to W, and we can write $A = L^2$ instead.

The area of a circle is given by the formula $A = \pi r^2$. The area of a triangle is given by $A = \frac{1}{2}bh$, where b is the length of the base, and h is the height.

Area is measured in square units. For example, if a rectangle is 8 meters in length and 5 meters in width, then its area is 40 square meters. This is written as 40 m^2.

Volume

The volume of a box is the product of its length, width, and height ($V = L \times W \times H$). If the box is a cube, then $L = W = H$, and we can write $V = L^3$ instead. The volume of a cylinder is equal to the area of the base times the height. The volume of a sphere of radius r is given by the formula $V = \frac{4}{3}\pi r^3$. Volume is measured in cubic units (e.g. cubic centimeters).

Conversion between units of volume sometimes causes students difficulty. How many cubic centimeters make up a cubic meter, if there are 100 centimeters in a meter? One might guess that there are 100 cubic centimeters in a cubic meter. But this cannot be right, for a cubic centimeter of water weighs only a gram, but a cubic meter of water weighs over a ton. The correct conversion is shown below.

$$1\ \text{m}^3 = (100\ \text{cm})^3 = 100^3\ \text{cm}^3 = 1{,}000{,}000\ \text{cm}^3$$

Exercises

1. What is the perimeter of a standard $8\frac{1}{2}$ inch by 11 inch sheet of paper?
2. What is the text area of one side of a standard sheet of paper, if there is a three-quarter inch margin on each side?
3. How many square inches are in a square foot?
4. The dimensions of the iPod nano are $1.6 \times 3.5 \times 0.27$ inches. What is the volume in cubic centimeters? (1 inch = 2.54 cm)

Answers: 56 in, 66.5 in^2, 144, 24.8 cm^3