Lecture 1: Arithmetic Review

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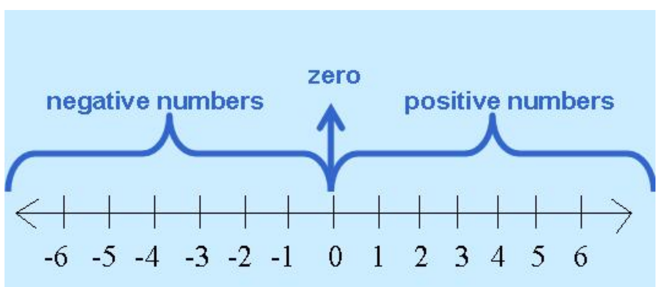
1 Integers

The integers $\mathbb{Z}$ consist of the positive whole numbers, the negative whole numbers, and 0.

$$\mathbb{Z} = \{\ \ldots,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ \ldots\ \}$$

We sometimes write positive numbers with a plus sign. For example, 3 and +3 are the same number. We also say that $+0 = -0 = 0$, although 0 is neither positive nor negative. Numbers with decimal or fractional parts, such as 2.45 and $\frac{3}{7}$, are not integers.

The integers can be represented as points on a **number line**. The positive integers lie to the right of 0, and the negative integers lie to the left of 0.



The **absolute value** of a number is its numerical value without regard to sign. For example, 3 is the absolute value of both 3 and -3. The absolute value of x is denote $|x|$.

Adding and subtracting integers

When adding two negative numbers, add their absolute values, and attach a negative sign.

$$(-3) + (-5) = -8$$

When adding a positive number and a negative number, subtract their absolute values, and attach the sign of the number having the larger absolute value.

$$(-3) + 5 = 2 \qquad (-5) + 3 = -2$$

When subtracting two numbers, reverse the sign of the second number, then add.

$$4 - 7 = 4 + (-7) = -3 \qquad 4 - (-7) = 4 + 7 = 11$$

The **additive inverse** (or **negative**) of a number x is denoted $-x$. The additive inverse of $-x$ is $-(-x) = x$. When a number and its negative are added, the result is 0.

$$x + (-x) = 0$$

Adding zero to a number does not change its value.

$$x + 0 = 0 + x = 0$$

Multiplying and dividing integers

A **product** is the result of multiplying two or more numbers. A **quotient** is the result of dividing two numbers.

The product of two numbers can be represented in many different ways. Here are some ways of denoting the product of 5 and 7:

$$5 \times 7 \qquad 5 \cdot 7 \qquad 5(7) \qquad 5 * 7$$

The product of two numbers with the same sign is positive, and the product of two numbers with different signs is negative. The same rules hold for the quotient of two numbers.

$$\begin{array}{llll} 3 \times 4 = 12 & 3 \times (-4) = -12 & (-3) \times 4 = -12 & (-3) \times (-4) = 12 \\ 8 \div 2 = 4 & (-8) \div 2 = -4 & 8 \div (-2) = -4 & (-8) \div (-2) = 4 \end{array}$$

Multiplying a number by 1 does not change its value. Multiplying a number by -1 reverses its sign.

$$6 \times 1 = 6 \qquad 6 \times (-1) = -6 \qquad (-5) \times (-1) = 5$$

The product of any number with zero is zero. If zero is divided by a nonzero number, the result is zero. However, division by zero is undefined.

$$0 \times (-4) = 0 \qquad 0 \div (-4) = 0 \qquad 6 \div 0 = \textit{undefined}$$

Exercise: Evaluate each expression.

$$(-3) - (-5) \qquad -18 + 9 - (-3) \qquad (-2)(-5) \qquad 442 \times 1863 \times 0 \qquad 0 \div 9 \qquad 9 \div 0$$

Answers: 2, -6, 10, 0, 0, undefined

2 Fractions and mixed numbers

The real numbers include the integers, but they also contain fractional numbers and decimals.

A **fraction** is a quotient or ratio of two positive integers. The first number is called the **numerator** and the second number is called the **denominator**. Fractions can be written in many different ways, for example:

$$\frac{5}{8} \qquad 5/8 \qquad 5:8 \qquad 5 \div 8$$

An integer can be expressed as a fraction having a denominator of 1.

$$n = \frac{n}{1} \qquad 23 = \frac{23}{1}$$

A fraction is **proper** if the numerator is less than the denominator. Otherwise it is **improper**. For example, 7/9 and 2/4 are proper fractions, but 6/6 and 7/3 are improper fractions.

Two fractions are **equivalent** if they represent the same value. For example, 1/2 and 2/4 are equivalent fractions. Multiplying the numerator and denominator of a fraction by the same nonzero number does not change the value of the fraction.

Examples:

$$\frac{7}{8} = \frac{7 \cdot 5}{8 \cdot 5} = \frac{35}{40}$$

$$\frac{30}{50} = \frac{30 \div 10}{50 \div 10} = \frac{3}{5}$$

A fraction is **reduced** if its numerator and denominator do not have a common factor. Every fraction is equivalent to a reduced fraction. For example, 6/9 is not reduced, because the numerator and denominator are both divisible by 3. Dividing the numerator and denominator by 3 gives the reduced fraction 2/3.

Adding fractions

To add two fractions with **like** denominators, just add the numerators.

$$\frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

To add two fractions with **unlike** denominators, change the fractions to equivalent fractions with like denominators, then add the numerators.

$$\frac{1}{3} + \frac{2}{5} = \frac{5}{15} + \frac{6}{15} = \frac{11}{15}$$

$$2 + \frac{2}{3} = \frac{2}{1} + \frac{2}{3} = \frac{6}{3} + \frac{2}{3} = \frac{8}{3}$$

You can also use the shortcut formula $\frac{A}{B} + \frac{C}{D} = \frac{AD + BC}{BD}$. *Example:*

$$\frac{2}{7} + \frac{5}{8} = \frac{2 \cdot 8 + 7 \cdot 5}{7 \cdot 8} = \frac{16 + 35}{56} = \frac{51}{56}$$

Subtracting fractions

To subtract two fractions with **like** denominators, just subtract the numerators.

$$\frac{5}{9} - \frac{1}{9} = \frac{4}{9}$$

To add two fractions with **unlike** denominators, change the fractions to equivalent fractions with like denominators, then subtract the numerators.

$$\frac{7}{9} - \frac{3}{4} = \frac{28}{36} - \frac{27}{36} = \frac{1}{36}$$

You can also use the shortcut formula $\quad \dfrac{A}{B} - \dfrac{C}{D} = \dfrac{AD - BC}{BD}.$

$$\frac{3}{7} - \frac{1}{3} = \frac{3 \cdot 3 - 4 \cdot 2}{4 \cdot 3} = \frac{9 - 8}{12} = \frac{1}{12}$$

Multiplying fractions

To multiply two fractions, multiply the numerators, and multiply the denominators. You should reduce the result.

$$\frac{A}{B} \times \frac{C}{D} = \frac{AC}{BD} \qquad \frac{3}{5} \times \frac{4}{7} = \frac{12}{35}$$
$$\frac{4}{7} \times \frac{5}{6} = \frac{20}{42} = \frac{10}{21}$$

If a numerator and denominator have a common factor, then you may cancel the factor before multiplying.

$$\frac{8}{15} \cdot \frac{7}{12} = \frac{{}^{2}\cancel{8}}{15} \cdot \frac{7}{\cancel{12}_{3}} = \frac{14}{45}$$

To multiply a fraction by an integer, simply multiply the numerator by the integer.

$$\frac{3}{19} \cdot 6 = \frac{3}{19} \cdot \frac{6}{1} = \frac{18}{19}$$

Reciprocals and division

The **reciprocal** (or **multiplicative inverse**) of a fraction $\dfrac{A}{B}$ is $\dfrac{B}{A}$. The reciprocal of A is $\dfrac{1}{A}$. The product of a number and its reciprocal is 1.

$$273 \cdot \frac{1}{273} = 1 \qquad \frac{123}{456} \cdot \frac{456}{123} = 1$$

To divide two fractions, multiply the first fraction by the reciprocal of the second fraction.

$$\frac{8}{15} \div \frac{4}{5} = \frac{8}{15} \cdot \frac{5}{4} = \frac{40}{60} = \frac{2}{3}$$

Mixed numbers

A mixed number is the sum of an integer and a proper fraction, but it is written without the plus sign. For example, $33\frac{1}{3}$ is a mixed number, and it means $33 + \frac{1}{3}$. The notation $33\frac{1}{3}$ should be avoided when doing algebra, because it is easily confused with $33 \cdot \frac{1}{3}$.

To convert a mixed number to an improper fraction, just perform the addition.

$$4\,\frac{5}{7} = \frac{4}{1} + \frac{5}{7} = \frac{28}{7} + \frac{5}{7} = \frac{33}{7}$$

To convert an improper fraction to a mixed number (with a calculator):

1. Divide the numerator by the denominator.
2. Subtract the integer part of the quotient.
3. Multiply the result by the denominator.

Examples: Convert 169/25 and 289/40 to mixed numbers.

$169 \div 25 = 6.760$	$6.760 - 6 = 0.760$	$0.760 \times 25 = 19$	$169/25 = 6\ 19/25$
$289 \div 40 = 7.225$	$7.225 - 7 = 0.225$	$0.225 \times 40 = 9$	$289/40 = 7\ 9/40$

3 Decimals

A **decimal fraction** (or **decimal** for short) is a number that looks like 123.4567. The period in the middle is called the **decimal point**. The part to the left of the decimal point is called the **integer part**, and the part to the right is called the **fractional part**.

The following examples shows how to convert a decimal to a fraction. Note that the number of zeros in the denominator is equal to the number of digits after the decimal point.

$$\begin{aligned}
123.4567 &= 123 + \frac{4}{10} + \frac{5}{100} + \frac{6}{1000} + \frac{7}{10000} \\
&= 123 + \frac{4000}{10000} + \frac{500}{1000} + \frac{60}{10000} + \frac{7}{10000} \\
&= 123\,\frac{4567}{10000} \\
14.921 &= 14\,\frac{921}{1000}
\end{aligned}$$

It is easy to convert a fraction to a decimal with a calculator. Simply perform the division, and the calculator will display the result as a decimal, rounded to the number of digits that can be displayed. Sometimes one wants to convert a rounded decimal back to the fraction that generated it. To do this, we subtract the integer part, take the reciprocal, and repeat these two steps until an integer value is obtained. Once an integer value is reached, we can reverse the steps to recover the original fraction.

Example: Convert 1.448275862 to a fraction.

Solution:

	integer part	fractional part	reciprocal
1.448275862	1	0.448275862	2.230769231
2.230769231	2	0.230769231	4.333333329
4.333333329	4	0.333333329	3.000000039

The last number is very close to 3. So we round it to 3, and then reverse the process.

$$\begin{array}{rclcl} 4.333333329 & = & 4+1/3 & = & 13/3 \\ 2.230769231 & = & 2+3/13 & = & 29/13 \\ 1.448275862 & = & 1+13/29 & = & 42/29 \end{array}$$

You can convert decimals to fractions with the TI-83 Plus by pressing [MATH] [ENTER], provided that the number is entered with full precision and the denominator is less than 10000.

4 Percentages

Percent means *per hundred*, and it indicates a fraction whose denominator is 100. Thus, 14 percent means 14 parts per hundred, or 14/100, or 0.14. We write 14 percent as 14% for short.

To convert a percentage to a fraction, just put the number over 100. To convert a percentage to a decimal number, shift the decimal two places to the left.

To find a percentage of a number, just multiply the number by the percentage (expressed as a decimal or fraction). For example, 25% of 60 is the same as $(0.25) \times 60$, or 15.

To find the percentage increase or decrease in a quantity, divide the change in the quantity by the original amount.

If a quantity Q is increased by $x\%$, then the new quantity is $(1 + x/100)Q$. If a quantity Q is decreased by $x\%$, then the new quantity is $(1 - x/100)Q$.

We give several typical examples of calculations involving percentages.

- What is 15% of 200? $0.15 \times 200 = 30$.
- What percentage of 250 is 60? $60 \div 250 = 0.24 = 24\%$.
- The price of a stock increased from 50 to 58. What was the percentage increase?
  (58-50)/50 = 0.16 = 16%
- A calculator costs \$70. If the price increases by 5%, what is the new price?
  $1.05 \times \$70 = \$10.35$
- A pair of designer jeans costs \$150. If the price is reduced by 30%, what is the new price?
  $0.70 \times \$150 = \$105$
- A restaurant bill is \$60.03, including a 15% tip. What is the cost of the meal, excluding the tip?
  $60.03 \div 1.15 = \$52.20$

5 Exponents

The notation b^n means that b is multiplied by itself n times, and we say that b is raised to the n^{th} power. The superscripted number n is called the **exponent**, and b is called the **base**.

$$5^1 = 5 \qquad 5^2 = 5 \times 5 = 25 \qquad 5^3 = 5 \times 5 \times 5 = 125 \qquad 5^4 = 5 \times 5 \times 5 \times 5 = 625$$

We extend this definition to allow zero and negative exponents via $b^0 = 1$ and $b^{-n} = 1/b^n$. For example, $2^{-3} = 1/2^3 = 1/8$, and $(-3)^0 = 1$.

The most important properties of exponents are summarized below.

$$b^0 = 1 \qquad b^{-m} = \frac{1}{b^m} \qquad b^m \cdot b^n = b^{m+n} \qquad \frac{b^m}{b^n} = b^{m-n}$$
$$(b^m)^n = b^{mn} \qquad (ab)^n = a^n \cdot b^n \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

It is important to distinguish between $(-b)^n$ and $-b^n$. The first expression means that $-b$ is raised to the n^{th} power, and the second denotes the negative of b^n. For example, $(-3)^2 = (-3)(-3) = 9$, but $-3^2 = -(3)(3) = -9$. If n is even, then $(-b)^n = b^n$, but if n is odd then $(-b)^n = -b^n$. As a special case, $(-1)^n = 1$ when n is even, and $(-1)^n = -1$ when n is odd.

Exercise: Simplify each expression using the rules of exponents.

$$2^{12} \cdot 5^{12} \qquad \frac{2^{20}}{2^{18}} \qquad (-1)^{33} \qquad 2^{12} \cdot 2^{-9} \qquad (-3)^0 \qquad (10^8)^{-3}$$

Answers: 10^{12}, 4, -1, 8, 1, 10^{-24}

6 Square roots and nth roots

Square roots. A **square root** of a number is one of its two equal factors. For example, 5 is a square root of 25 because $5 \times 5 = 25$. Every positive number has two square roots; one is positive, the other negative. The symbol $\sqrt{x}$ denotes the **nonnegative** square root only.

When we speak of "the" square root of a number, we mean the **nonnegative** square root. Thus, the square root of 4 is 2, not -2.

Definition of nth root. Let a and b be real numbers and let $n \geq 2$ be a positive integer. If $a = b^n$ then b is an **nth root of a.** If $n = 2$, then the root is a **square root.** If $n = 3$, then the root is a **cube root.**

Existence of nth roots. Let a be a real number, and let $n \geq 2$ be a positive integer.

1. If n is odd, then a has exactly one real nth root, which is written as $\sqrt[n]{a}$. This root has the same sign as a.
2. If n is even and a is positive, then a has exactly two nth roots, which are additive inverses of each other. The positive nth root is $\sqrt[n]{a}$ and the negative root is $-\sqrt[n]{a}$.
3. If n is even and a is negative, then a does not have an nth root.
4. 0 has exactly one nth root, namely 0. We write $\sqrt[n]{0} = 0$.

Properties of Radicals

$$\sqrt{x} \cdot \sqrt{x} = x$$
$$\left(\sqrt[n]{x}\right)^n = x$$
$$\sqrt{x} \cdot \sqrt{y} = \sqrt{x \cdot y} \quad \text{(Product rule)}$$
$$\frac{\sqrt[n]{x}}{\sqrt[n]{y}} = \sqrt[n]{\frac{x}{y}} \quad \text{(Quotient rule)}$$

Fractional exponents

Square roots and nth roots can be expressed using fractional exponents. This is useful the laws of exponents also apply to fractional exponents

$$\begin{aligned} x^{1/2} &= \sqrt{x} \\ x^{1/n} &= \sqrt[n]{x} \\ x^{m/n} &= \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m \end{aligned}$$

7 Order of operations

When evaluating an expression, it is important to apply the operators in the correct order. For example, consider the expression $2 + 3 \times 4$. If you perform the addition first then you get $5 \times 4 = 20$, but if you multiply first then you get $2 + 12 = 14$. We cannot have two different values for the same expression, so we must agree on the order in which operations are to be performed.

The order of operations is as follows.

1. Apply all exponents.
2. Perform all multiplications and divisions.
3. Perform all additions and subtractions.

We say that exponentiation has higher priority than multiplication, and multiplication has higher priority than addition. Multiplication and division have the same priority. Addition and subtraction also have the same priority. Operations with the same priority should be performed from left to right.

For example, let us evaluate

$$3 + 5 \times 2^3 - 4 \times 6.$$

First we calculate $2^3 = 8$, because exponentiation has the highest priority.

$$3 + 5 \times 8 - 4.$$

Next, we perform the multiplication $5 \times 8 = 40$.

$$3 + 40 - 4$$

Finally, we perform the additions and subtractions from left to right.

$$3 + 40 - 4 = 43 - 4 = 39$$

The order of operations can be controlled by means of parentheses and other grouping symbols. If parentheses are present, then the expression inside the parentheses must be evaluated first. For example, to evaluate $4 \times (2 + 3)$, we must perform the addition first, because it is enclosed by parentheses.

$$4 \times (2 + 3) = 4 \times 5 = 20$$

Sometimes, parentheses occur inside parentheses. This is called nesting. If this happens, then the innermost expression, which is surrounded by the most parentheses, is to be evaluated first.

Example:

$$
\begin{aligned}
3(4+5\times(8-6))+7 &= 3(4+5\times 2)+7 \\
&= 3(4+10)+7 \\
&= 3(14)+7 \\
&= 42+7 \\
&= 49.
\end{aligned}
$$

Horizontal fraction bars serve as grouping symbols. The numerator and denominator should be treated as if they were enclosed by parentheses. The horizontal bar over a square root is also a grouping symbol.

Example:

$$
\begin{aligned}
3+\frac{2+3\times 4}{5-3} &= 3+(2+3\times 4)\div(5-3) \\
&= 3+(2+12)\div(5-3) \\
&= 3+14\div 2 \\
&= 3+7 \\
&= 10.
\end{aligned}
$$

When we write a negative number, such as -5, the minus sign is treated as subtraction when determining the order of operations. For example, $-5^2 = -25$ because the exponentiation is performed first. On the other hand, $(-5)^2 = (-5)\times(-5) = 25$.

Exercises: Evaluate each of the following expressions without using a calculator. Then use a calculator to check your work.

1. $10-6+4$
2. $3\times 2+4\times 4-5^0$
3. $(1+4-8)-(3-7-7)$
4. $-4^2-\dfrac{2\cdot 4-\frac{3}{5}\cdot 15}{6+12/3-8}$

Answers: 8, 21, 8, -15.5